Historical Notes
For the Calculus Classroom

V. Frederick Rickey

Igitur eme, lege, fruere.
-- Copernicus, 1543

Prepared for the
Institute in the History of Mathematics and Its Use in Teaching
The American University, 3 - 21 June, 1996

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What to Say the First Day

> Calculus is the culmination of a dramatic intellectual struggle which has lasted for over 2500 years and has proved itself to be the greatest achievement of western civilization.
>
> Richard Courant (1888–1972)

What is said on the first day of a course is tremendously important, for it sets the tone of the whole term. This is our opportunity to show our new students that this will be a rich and exciting course, one filled with interesting and valuable ideas. During the first class, students should get the feel of the whole course, what the subject of calculus is about and why it is important. Not just important historically, or important for society, but important and useful to them in the careers that they intend. Students should be given a preview of the main ideas that they will encounter as well as a glimpse of some of the interesting problems that we will do using the calculus. The first class is not the time to discuss details, much less to begin to present the nitty-gritty of the material, but the time to present a grand sweep of ideas that gives the flavor of what we are about to undertake. Since the calculus has made many important contributions to the betterment of mankind, it is reasonable to expect that it will continue to do so in the future. This is why the faculty in many disciplines have required that their students learn some calculus. My aim here is to show that history can be used to get the students off on the right foot.

In addition, we teachers are aware that there are a number of organizational topics that must be discussed on the first day, but I eliminate as many as possible by passing out a detailed syllabus, telling the students to read it carefully and then letting them ask questions about it during the second class. Even so, it is worth the time and effort to give them a pep talk about study skills. But they have heard that before, so I recast the information in terms of historical examples to illustrate the points that I want to make about working hard.

What follows is a pseudo-transcript of one of my typical first day calculus classes together with reflections for teachers about why I do what I do.

What is Calculus?

One thing that I have noted is that calculus books never answer the question (or even raise it) of what calculus is. This seems to me to be a major flaw in our textbooks. At the beginning of any course we have a real obligation to make some attempt to explain to the students what we are up to, especially since we are going to ask them to devote two to four semesters of their time to Calculus. We should not just begin with a dozen definitions and Theorem 1. Admittedly it is very difficult to explain to a neophyte what the calculus is all about. Perhaps this explains why our textbook authors avoid the question entirely.

Rather that launch into a detailed discussion about the invention of the calculus, I shall report here what I tell my students. The idea is to pare the story to the essentials, keeping only those facts which can benefit them at their present level of understanding. As the course continues, I fill in more of the details as the occasion arises.

Calculus is the science of change, the study of things that move. It was created by Isaac Newton and Gottfried Leibniz (and at this point I put their pictures on the overhead) in the seventeenth century. They and, consequently, we use the word "calculus" in its etymological sense of "to pebble," that is, to compute, for it is an almost algorithmic method of solving problems of certain types. They were interested in studying curves and trying to discover their most interesting properties. We study functions because they can be used to represent curves and are important in other ways too. They and we are especially interested in properties such as maxima and minima and areas (all the time I am drawing pictures for ~~them~~ the students, for I want to motivate the preliminary study of analytic geometry and functions).

It is true that Newton and Leibniz had numerous predecessors who solved problems using what we see today to be the calculus. For example, the Greeks knew how to find tangent lines to conics, Kepler solved maxima problems dealing with wine barrels, and Stevin solved the problem of force on a dam. But their methods were ad hoc, and not subject to generalization.

These two problems, the max-min problem and the area problem give rise to the two branches of the Calculus, the differential and the integral calculus. The max-min problem reduces to finding the tangent line to a curve at a given point, for if we can find the places where the tangent line is horizontal, then we can find maxima and minima. This problem seems unrelated to the problem of finding the area under a curve, but it is not unrelated. One of the great discoveries of Newton and Leibniz independently is that these two problems are intimately connected. This is such a surprising and important relationship that we call it the Fundamental Theorem of Calculus. Newton and Leibniz are credited with the discovery of the calculus because they found and utilized this relationship, because they saw that the problem was to find the function that is the integral or derivative of the given function, and also because they found algorithmic ways of solving these problems.

The student's initial reaction is that they are not interested in these two problems and their intimacies. So I respond that—and this is one of the wonders of mathematics and the primary benefit of abstraction—our techniques for solving these problems apply to many other situations, in particular to real world problems.

Mathematics is useful.

I say again (to use a phrase I learned at West Point): Please note carefully that I did *not* say that Newton and Leibniz invented the calculus in order to apply it to the real world, but rather that it turned out (rather quickly in fact) that it could be applied to real world problems. Calculus was initially done as a piece of pure mathematics motivated by internal considerations that the mathematical community had been dealing with for a long time. I sincerely wish that they had had applications as their primary concern—for we and our students certainly should have applications of the calculus as our primary concern—but Newton and Leibniz did not. We, as teachers, and certainly not as historians, should not distort the truth on so important a matter as this.[1]

Let me relate one of the earliest and one of the most important applications of the calculus. Robert Hooke, who was sent by the Royal Society of London, asked Newton which physical law determined the gravitational attraction between the earth and the planets. He responded that gravitation was inversely proportional to the square of the distance between the objects. Hooke asked if he could prove this and Newton responded that he had, but he couldn't find his proof. The result of this conversation was the most famous scientific book ever written, the *Philosophia naturalis principia mathematicae* (1687)

Then I begin a litany of applications of the calculus: Calculus explains everything from how the solar system works (as Newton discovered) to the shape of the cable on a suspension bridge, to how much our trachea contracts when we cough to how much of the surface area of a hard disk should be used for storing information (my in class list is longer). In fact, in the three hundred years after Newton and Leibniz, calculus provided the tool to solve a whole host of physical problems. So it is no exaggeration to say that the calculus has stood the test of time.

We are interested in the calculus today because it is still of value in attacking real world problems. It is true that most of the problems amenable to the calculus in the past have been from the physical sciences, but now the calculus has applications in a whole host of disciplines, including economics, sociology, biology, banking, etc. Usually, I ask a few students what their major is and then give them an example of how the calculus can be used in their field. This goes a long way to winning the class over to my viewpoint that this is going to be an interesting endeavor.

This beginning is intended to let the student know that we are going to study the calculus because it is useful. It shows that the instructor is willing to explain what the fundamental problems and ideas are and why they are important.

This claim of usefulness is a very easy claim to make, but we will have to demonstrate it throughout the course with a host of interesting examples if we are to convince our students. This is one place where I think that history of mathematics

[1] The myth that Newton created the calculus back on the farm so that he could do celestial mechanics and that he was primarily a physicist was created in a paper by Borris Hessen in 1931 entitled "The social and economic roots of Newton's Principia," pp. 149-212 in *Science at the Cross Roads*. This Marxist interpretation of Newton's work is flawed at almost every turn, but is still fascinating to read.

can really help. Historical examples are nice, because you can convincingly say that this particular piece of mathematics was once useful. But note that the mathematics preceded the applications. It can't be the other way around. I make this point to explain to the students why we do the mathematics first and that ~~are~~ our applications are quite simplistic at first. I support this by quoting Benjamin Franklin when he saw the first balloon launch. When he asked what good it was, he responded by asking of what use was a new baby.

Read Your Book.

Tolle lege, tolle lege.
St. Augustine[2]

After having spent more than half the class talking about the calculus and its history, there are a number of other points that I like to make. Most of them relate to study skills, an area where my students can use improvement. One point made in the "Hints for Success" portion of my syllabus is this faint echo of Augustine:

> Read the book. Read it before class, then again before attempting the homework, and again after doing the problems. When you read the book do it slowly and carefully, with pencil in hand. Pay careful attention to the definitions and examples. Omit the proofs on a first reading.

But I want to reinforce this admonition on the first class day. Since, in describing what calculus is, I have mentioned ~~to~~ analytic geometry, it fits in nicely to present (on an overhead) this quotation from Newton describing how he began his study of mathematics:

> Took Descartes's Geometry in hand, tho he had been told it would be very difficult, read some ten pages in it, then stopt, began again, went a little farther than the first time, stopt again, went back again to the beginning, read on til by degrees he made himself master of the whole, to that degree that he understood Descartes's Geometry better than he had done Euclid.[3]

Newton told this story to Abraham DeMoivre (1667–1754) late in ~~his~~ Newton's life about how he studied mathematics. He read Descartes' geometry (not the hundred page original French of 1637, but the nine-hundred page second Latin edition of 1659-1661). This was one of the very best mathematics books of the day and a fairly new and quite advanced book at that time. But most importantly, look at how he described his progress: Read a few pages, and then read them over again, and continued that cycle until the material had been mastered. Now, Newton was one of the kids in the bright group. If he had to read his book over and over and over again in order to understand it, then it behooves you (my dear students) to consider reading your book at least once.

If I just tell students to read their book, they probably will not listen to me. Perhaps this true story will convince them of the merit of careful reading.

Another remark that I make here is that they are going to have to read the book, I can not and will not do it for them. I try to minimize the amount of lecturing that I do and spend the class time more profitably interacting with students, answering their questions and helping them come to grips with the concepts.

Finally, a lesson is being planted for the future. It is advantageous to read high quality books, original sources, rather than spending ones time on secondary accounts. (possessive)

Do Your Homework.

When George Dantzig was a graduate student he arrived late for class one day. On the board were two problems. He presumed they were homework, copied them down, and went home and worked on them. Several days later he turned them in and apologized for taking so long. His busy teacher just threw them on the desk.

One Sunday morning six weeks later Dantzig was awakened by someone banging on his door. It was his teacher, Jerzy Neyman, and he was all excited: "I've just written an introduction to one of your papers. Read it so I can send it out right away for publication." It took Dantzig a few minutes to realize what Neyman was talking about: The problems were examples of problems that no

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2 "Take up, read! Take up, read!" is a literal translation of this line ~~form~~ from St. Augustine (354-430), *Confessions*, VIII, 12.

3 *The Mathematical Papers of Isaac Newton*, edited by D. T. Whiteside, vol. 1, p. 5–6. More information on Newton's readings is easily accessible in my "Isaac Newton: Man, Myth, and Mathematics," *The College Mathematics Journal*, vol. 18 (1987), pp. 362–389.

one had ever solved before; mathematicians call them "open problems."[4]

Years later the Reverend Schuler of the Crystal Cathedral in Los Angeles gave a sermon about Dantzig and the power of positive thinking: If Dantzig had known the problems were unsolved he would not have had such a positive attitude about solving them and probably would have quit too soon.[5]

That real Dantzig was my thesis advisor

Today Dantzig is a famous mathematician. His most often used result is the method of linear programming. He developed it during World War II to solve a number of complex scheduling problems for the military. Today it is used every day in the business world.

> Moral: Don't become discouraged while doing your homework. You can do it!
>
> P.S. Come to class on time.

I have the above story on a (far too crowded) overhead and when I used it at a class at West Point, I had to push it up so that the students could see the postscript. When I did, one cadet whispered to the officer next to her, "How did he know I was going to be late?" After learning of this, I always do it this way in class, for there is always someone who is late.

Besides making the point that it is important to do your homework, this story about Dantzig makes the point that problems — hard problems — are the essence of mathematics.

> Important though the general concepts and propositions may be with which the modern industrious passion for axiomatizing and generalizing has presented us, . . . nevertheless I am convinced that the special problems in all their complexity constitute the stock and core of mathematics; and to master their difficulties requires on the whole the harder labor.
>
> Hermann Weyl[6]

Visit During Office Hours

The times of my office hours are announced on my syllabus, but I want to make it clear that the following statement on the syllabus is not just window dressing.

> If you cannot come at these times please make an appointment for another time. You may catch me in my office many other times, but I only promise to be there at the previously stated times. You are encouraged to stop in to see me. It is part of your education.

I really want students to come see me if they are having difficulties or even if they just want to talk, but I also want them to prepare before they do. Consequently, I use the following quote from Leonhard Euler to make a point.

> I soon found an opportunity to be introduced to a famous professor Johann Bernoulli, whose good pleasure it was to advance me further in the mathematical sciences. True, he was very busy and so refused flatly to give me private lessons; but he gave me much more valuable advice to start reading more difficult mathematical books on my own and to study them as diligently as I could; if I came across some obstacle or difficulty, I was given permission to visit him freely every Saturday afternoon and he kindly explained to me everything I could not understand, which happened with such greatly desired advantage that whenever he had obvitiated one difficulty for me, because of that ten others disappeared right away, and this, undoubtedly, is the best method to succeed in mathematical subjects.[7]

Naturally, I mention that the Swiss mathematician Johann Bernoulli was one of the first to master the new calculus of Leibniz and was thus tremendously influential in informing interested individuals on the continent about the power of the calculus. Euler was his most important student and became the best mathematician of the eighteenth-century and the most productive of all

[4] One reason for making a comment such as this is to acquaint the student with the vocabulary of the profession. They should learn how mathematicians interact, that they do things besides teach class, and, perhaps most importantly, that not all mathematics has been done long ago. There are plenty of open problems and the best students can solve some of them.

[5] This story comes from an "An interview with George B. Dantzig: The father of linear programming," by Donald J. Albers and Constance Reid, *The College Mathematics Journal*, 17(1986), 292–314. He is the son of Tobias Dantzig, author of *Number, the Language of Science (1930)*.

[6] Hermann Weyl, *The Classical Groups*, 1939, p. xi-xii.

[7] Autobiography of 1767; combined quotes from DSB IV, 468 and Truesdell, xii.

time. While giving a bit of biographical information about Euler, I show a picture of him on the overhead. One thing that calls out for comment then is his blindness. I comment, matter of factly, that this did not hinder his mathematical publication. Without saying anything this informs the students that mathematics is a profession that even those with the handicap of blindness can take up.

I encourage my students to make a list of the questions they have before they come to my office (and, in fact, before they come to class). Not just a list of the problems they can't do; I encourage them to write out questions that ask about the mathematical concepts. I am honest with them about my motivation: doing this will help them understand the material better. This is hard for them at first, but they come to see the wisdom in it, partly because of the advice that Euler has given.

Sometimes I use a different quotation, this one from Isaac Barrow. It was originally written in Latin, and it is not any clearer in Latin. But it does make the students take note.

> If it be then your Pleasure, ye Lovers of Study, come always; be not restrained through any Fear, or retarded too much by Modesty, what you may do by your Right, you shall make me do willingly, nay gladly and joyfully. Ask your Questions, make your Enquiries, bid and command; you shall neither find me adverse nor refractory to your Commands but officious and obedient. If you meet with any Obstacles or Difficulties, or are retarded with any Doubts while you are walking in the cumbersome Road of this Study of *Mathematics*, I beg you to impart them, and I shall endeavor to remove every Hindrance out of your Way to the best of my Knowledge and Ability.

Isaac Barrow
"Prefatory Oration," 1664

Yes, Calculus is Hard

Albert Einstein once said:

> I have little patience with scientists who take a board of wood, look for its thinnest part, and drill a great number of holes where drilling is easy.

Well we are not going to make that mistake. We will attack hard problems for that is where the real meat is. I tell the students that they should not be be discouraged by hard problems, for it is by attacking them that we learn. This little poem by Piet Hein makes my point nicely.

A MAXIM FOR VIKINGS

Here is a fact
that should help you fight
a bit longer:

Things that don't actually kill you outright
make you stronger.

A Function is Not a Set of Ordered Pairs

One of the travesties of the new math is the ordered pair definition of a function: a function is a set of ordered pairs no two of which have the same first element. Students find the notion unintuitive, unmotivated, and unattractive—and they are perfectly justified. This is a fine example of abstraction for its own sake. The definition is never used (except for some concocted problems in the section where the definition was introduced), and that is perfectly understandable, because it is useless.

The only time that I can think of when the ordered pair definition has some slight pedagogical advantage is when one is discussing inverse functions. The only place where it is really necessary is when one is building models of set theory, but few mathematicians ever do this, and then not until they are graduate students.

To understand this definition it is necessary to examine its philosophical and historical roots. Gottlob Frege (1848–1925), a distinguished German mathematician, whose views on the foundations of mathematics and language were significantly ahead of his time, realized (as did Dedekind, Heine, Meray, and Weierstrass) that the natural numbers lacked a firm foundation. His solution to this problem led him to adopt the philosophical position that all of mathematics could be built up from logic alone. Later, because of the work of Bertrand Russell (1872–1970), it was realized that set theory was also necessary to complete this program. The modified position—that all of mathematics can be built up from set theory and logic—is referred to as the Frege-Russell Thesis.

Needless to say, many details needed to be supplied to support this thesis (since it is a philosophical position, it is difficult to see how it could be proved). One of the most important details in this program was to define the concept of a function in terms of sets. As often happens, several mathematicians solved the problem almost simultaneously.

In 1913, the eighteen year old Norbert Wiener (1894–1964) received his Ph.D. from Harvard for a dissertation comparing the logical system of Alfred North Whitehead (1861–1947) and Russell with that of Ernst Schröder (1841–1902). Under a traveling fellowship granted by Harvard, Wiener then went to England to study with Russell. A consequence of this was his 1914 paper "A simplification of the logic of relations,"[1] In this paper Wiener reduced the theory of relations to the theory of classes by providing the following definition of ordered pair, which we translate from the notation of Whitehead and Russell:

$$\langle x, y\rangle = \{\,\{\{x\}, \emptyset\}, \{\{y\}\}\,\}.$$

What Wiener did was to revert to Schröder's treatment of relations as sets of ordered pairs. He was aware of the ontological significance of what he had done:

> The complicated apparatus ... is simply and solely devised for the purpose of constructing a class which shall depend only on an ordered pair of values of x and y, and which shall correspond to only one such pair. The particular method selected of doing this is largely a matter of choice.

Wiener realized that the important thing about his definition is that it be sufficient to prove

$$\langle a, b\rangle = \langle c, d\rangle \quad \Rightarrow \quad a = c \text{ and } b = d.$$

At approximately the same time Felix Hausdorff (1868–1942), in the second book about set theory, provided the following definition of ordered pair[2]

$$\langle x, y\rangle = \{\,\{1, x\}, \{2, y\}\,\},$$

where 1 and 2 are distinct sets and x and y are distinct from each of them. This definition only appears in the first edition of Hausdorff's work, probably because he felt (or had been told) it would not work. Curiously, if the restriction that x and y be

[1] *Proceedings of the Cambridge Philosophical Society*, vol. 17(1912–1914), pp. 387–390; reprinted in J. van Heijenoort (ed.), *From Frege to Gödel*, pp. 224–227.

[2] *Grundzüge der Mengenlehre*(1914), p. 34. This note is not in the Dover reprint of the third edition, entitled simply *Mengenlehre*.

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distinct from 1 and 2 is dropped, the definition works just fine.

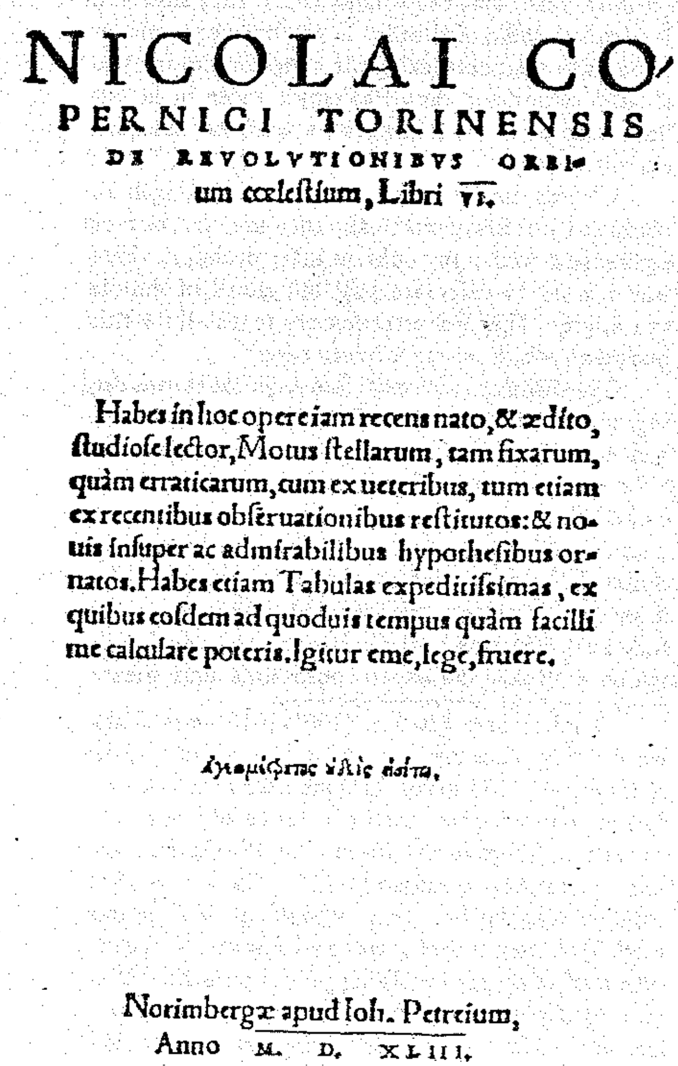
In 1921 Kazimierz Kuratowski (1896–1980) published the definition of ordered pair that has been universally adopted. It appears on the last page of his paper on the notion of order in set theory[3]

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

Kuratowski points out that his definition is simpler than Hausdorff's, since Hausdorff required that the sets a and b be distinct from 1 and 2. This argument is wrong, since Hausdorff's definition is sufficient to prove the required theorem without this restriction.

Kuratowski's definition of the ordered pair in terms of sets arose because of criticism by Jan Łukasiewicz (1878–1956) of a textbook by Stanisław Zaremba (1863–1942)[4] In 1912 Zaremba, a professor at the Jagellonian University in Cracow, published *Arytmetyka teoretyczna (Theoretical Arithmetic)*. The book was well received by the mathematical community, but in 1916 Łukasiewicz published a stinging attack on the logical underpinings of the work.[5]

This is my favorite example of how a knowledge of the history of mathematics indicates what we should not teach. Anyone who truly believes that a function really is a set of ordered pairs with some other funny property has such a perverted view of what mathematics is all about, that they probably should not be turned loose in the classroom. The ordered pair definition of a function should only be presented in a course on the foundations of mathematics, and then the historical background must be given so that the definition makes sense.



NICOLAI CO
PERNICI TORINENSIS
DE REVOLVTIONIBVS ORBI-
um cœlestium, Libri VI.

Habes in hoc opere iam recens nato, & ædito,
studiose lector, Motus stellarum, tam fixarum,
quàm erraticarum, cum ex ueteribus, tum etiam
ex recentibus obseruationibus restitutos: & no-
uis insuper ac admirabilibus hypothesibus or-
natos. Habes etiam Tabulas expeditissimas, ex
quibus eosdem ad quoduis tempus quàm facilli
me calculare poteris. Igitur eme, lege, fruere.

Ἀγεωμέτρητος οὐδεὶς εἰσίτω.

Norimbergæ apud Ioh. Petreium,
Anno M. D. XLIII.

[3] "Sur la notion de l'ordre dans le Théorie des Ensembles," *Fundamenta Mathematicae*, vol. 2(1921), pp. 161-171.

[4] K. Kuratowski, *A Half Century of Polish Mathematics*(1980), p. 24.

[5] Portions of "O pojęciu wielkości," *Przegląd Filozoficzny*, vol. 19(1916), pp. 1–70 are translated as "On the concept of magnitude," in *Jan Łukasiewicz, Selected Works* (1970), pp. 64–83.

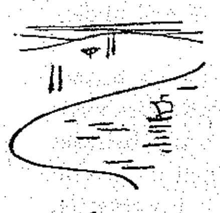
Etymology of the word 'sine'

Trigonometry has its origin in the astronomy of the Babylonians of the third-century BC and was then taken up by the Greeks. The first trigonometric table was constructed by Hipparchus about 150 B.C., who is accordingly known as the father of trigonometry.[1] This was a table of lengths of chords subtended by arcs in a fixed circle, for this is the natural length to consider as corresponding to a given arc. While this table of Hipparchus has been lost, as has the table of Menelaus (c. A.D. 100), that of Ptolemy (A.D. 150) survives. It also is a table of chords. If you have any doubts about the astronomical roots of trigonometry you should remember that Ptolemy's table is in the first chapter of his *Almagest*, a work on theoretical astronomy.

Chords may be the most natural length to compute, but mathematically they are not the most convenient to us. In problem after problem, there was a need to calculate half the chord of double the angle. This led astronomers to tabulate this quantity, which is our modern sine.[2]

The Hindu mathematician Āryabhata was dealing with half-chords by AD 500. He called these half-chords *ardha-jya* in Sanscrit and frequently abbreviated *jya* (chord). From this word the Arabs coined the technical term *jiba* meaning half-chord. Because of the fact that vowels were frequently omitted the abbreviation *jb* was misconstrued by later Arabic writers, who were also unaware of the fact that *jiba* was a neologism, as the word *jaib*, which contains the same consonants and means "cave" or "bay." [cf., Katz, p. 201]

Scholars in Europe in the twelfth century were fascinated by the work of the Arabs on algebra and trigonometry and so naturally were anxious to see translations into Latin. Around 1150 Robert of Chester and Gerhardo of Cremona translated several trigonometrical works into Latin. They also confused the Arabic words *jiba* and *jaib* which they translated as *sinus*, the Latin word for the fold of a toga about the breast or for a hollow or gulf (your sinuses are those hollow places in your skull).[3]



sinus

sinus

This then is the origin of our word *sine*. It is due to the misunderstanding and mistranslation of

[1] Bartholomeo Pitiscus (1561-1613) coined the word "trigonometry," meaning the measurement of triangles, 1595 in his *Trigonometria: sive de solutione triangulorum tractatus brevis et perspicuus*, revised as *Trigonometriae sive de dimensione triangulorum libri quinque* (1600). There is a 1614 English translation. Interestingly, it was in this work that trigonometry was first applied to the solution of triangles involving points on earth. Previously trigonometry was only used to solve spherical and plain triangles involving astronomy. See Victor Katz, *A History of Mathematics. An Introduction* (1993), p. 367.

[2] See Katz, pp. 140-142.

[3] These delightful illustrations are from Hans H. Oerberg, *Lingua Latina secundum naturae rationem explicata*, Naturmethodens Sproginstitut (The Nature Method Institutes): Copenhagen, four volumes, third edition 1965, pp. 708 and 759 (in the fourth volume, although the volumes are paginated continuously). I recommend these volumes as a superb way to learn Latin. They are completely in Latin except for the copyright notice and they commence with a simple, but well illustrated, story about a family and progress through the grammar and vocabulary in small steps (the only dictionary I used was an English dictionary that contained etymologies). I owe a great debt to Magister Boleslaus Povsic who, after the first year, spoke only Latin in class. It was a wonderful way to learn the language.

This file, Sine-TheWord.tex, was typeset using TeX on May 29, 1996 at 0:56 A.M.

a technical term which was coined because Arabic had no word corresponding to the Hindu word for half-chord.

By happy accident the Latin meaning of fold or curve of a tunic took on a new meaning, but that took several centuries more. The first to sketch a sine curve was Giles Personne de Roberval in the seventeenth-century when computing the area under the cycloid, although he did not know what he had done. Thus begins the long process of abstraction from sines as lengths of half-chords to the sine as a function of an angle. This process culminated in the work of Euler in the eighteenth-century when he introduced the trigonometric ratios and the unit circle. It is only at this time that trigonometry takes on the form that is familiar to us today.

The Other Trigonometric Functions

The terms 'umbra recta' and 'umbra versa' were introduced for the tangent and cotangent in the twelfth-century by Gerard of Cremona while translating work from Arabic to Latin ('umbra' is the Latin word for shadow). The words 'tangent' and 'secant' were coined by the physician and mathematician Thomas Finck of Basel in 1583 in his *Geometria rotundi*, and adopted by Pitiscus.[4]

It was also in the twelfth-century that what we call the cosine was given a special name. Pitiscus also used the term 'sinus complementi' (sine of the complement) and then in 1620 Edmund Gunter (1581–1626) abbreviated this as 'co. sinus.' In 1658, John Newton shortened this to 'cosinus.' Gunter also coined the word 'cotangent.'

The secant and cosecant were almost unknown before the sixteenth-century.[5]

REMARK

There are many lessons to be learned from the history of trigonometry, but alas it is the least developed of all fields of elementary mathematics. The story told above varies in detail and emphasis from historian to historian and so should be understood as my interpretation of the facts. Suggestions for improvements would be welcomed.

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[4] Florian Cajori, *A History of Mathematics*, 1924, pp. 132 and 151.

[5] Vera Sanford (1891-1971), *A Short History of Mathematics* (1930), p. 298.

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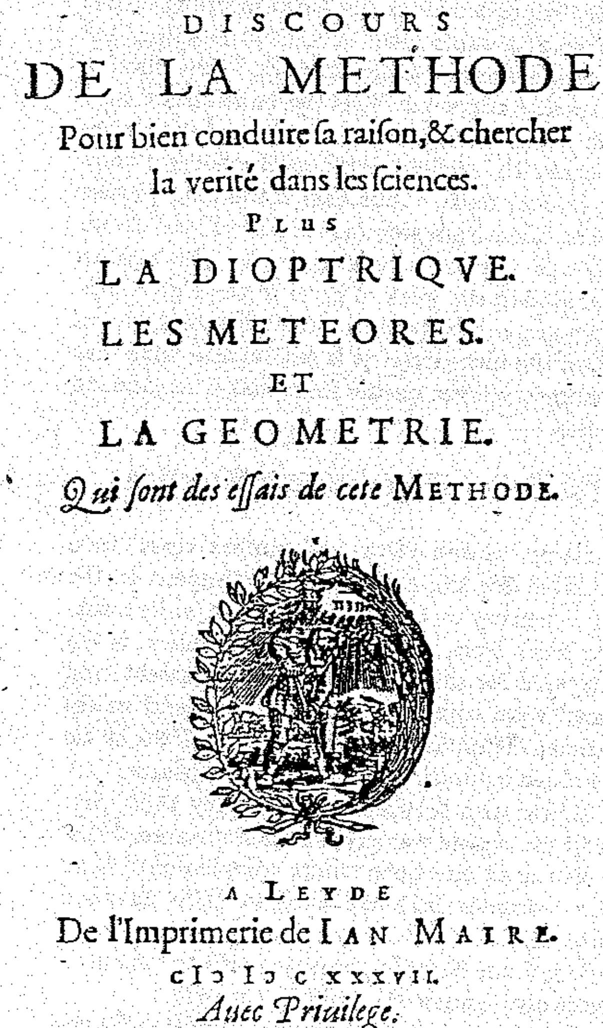
Hyperboloid of One Sheet



DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
CIƆ IƆ C XXXVII.
Auec Priuilege.

One of the most interesting scientific discoveries of all time was that of the telescope by the Dutch and its subsequent use and development by Galileo in the early seventeenth century. But it was not long before the defects of early telescope lenses were realized. Besides the technological problem of obtaining clear glass from which to make lenses, there was also the problem of chromatic aberration, the physical distortion caused by the different characteristics of the colored components of white light. Of mathematical interest was the problem of spherical aberration. This distortion stemmed from the fact that a spherical surface—and that was the only shape the lenses could then be given in practice—was not a theoretically correct form for a lens. With a spherical lens the incoming parallel rays of light did not focus at a point, but spread out along the axis of the lens.

Considering his use of the ellipse in describing the laws of planatary motion, it is not surprising that Johannes Kepler (1571–1630) recommended the use of lenses with hyperbolical surfaces. The advantages of elliptical, parabolic and hyperbolic lenses were generally accepted in the early seventeenth century, as is especially clear in the work of Descartes.

All are aware that Descartes was—together with Fermat—the founder of analytic geometry. The *Discours de la Méthode*, whose full title is "Discourse on the Method for Rightly Directing One's Reason and Searching for Truth in the Sciences," was published anonymously in 1637 (note the use of "open" Roman numerals on the title page), but it was an open secret that Descartes was the author. But not all today are aware that *La Géométrie* was only one of the three appendices of this book; the others dealt with optics (or more specifically with dioptrics) and metereology. The French original was translated into Latin, the *lingua franca* of the intellectual world, by Frans van Schooten (1615–1660) in 1649. This in turn was enlarged into a two volume heavily annotated work in 1659–1661 with commentaries by a number of others. It was this edition that influenced a generation of mathematicians including the young Newton. An English translation, *The Geometry of René Descartes* is available from Dover. This has the original French on facing pages so it is useful for making overheads to show the students the notation that he used. This is the earliest work that can reasonably be read today without dealing with archaic notation. There are many editions of the philosophical part on method, but there is only one translation of the whole and that is by Paul Olscamp, erstwhile philosopher at OSU and former president of BGSU. It should be consulted for the information on the conics which is in the first appendix and hence not in the Dover edition.

It is in the seventh discourse of the Optics, entitled "Of the Shapes that Transparent Bodies Must Have in Order to Divert Rays Through Re-fraction in Every Way That is Useful to Sight," that Descartes carefully points out the defects of spherical lenses and then recommends the use of parabolic and hyperbolic lenses. But he does much more; he gives proofs that such lenses have the desired optical properties.

In the tenth discourse, "On the Method of Cutting Lenses," Descartes suggests several methods for making hyperbolical lenses, but it seems clear that he has not built the machines that he describes (p. 173) and his tools proved ineffective.[1]

Another suggestion about constructing lenses whose cross sections are hyperbolic was made by Christopher Wren (1632-1723), who is best known today as the architect of such wonderful buildings at the Sheldonian Theater at Oxford and St. Paul's Cathedral in London. Wren was a student at Wadham College, Oxford where his talent for mathematical and scientific pursuits attracted attention. He graduated B.A. in 1651, M.A. in 1654 and stayed on as a fellow until he was appointed professor of astronomy at Gresham College in 1657. He held this post until 1661 when he was appointed Savilian professor of astronomy at Oxford, a post he held until 1673. Wren had an interest in architecture before 1663, for in that year he was commissioned to submit plans for the chapel of Pembroke College, Cambridge. It was the great London fire of 1666 that sealed his fate as an architect.[2]

Chief among Wren's contributions to mathematics was his rectification of the cycloid, but that is the topic of another note. In the June 21, 1669 issue of the *Philosophical Transactions of the Royal Society of London* Wren published a one page note on "The Generation of an Hyperbolical Cylindroid demonstrated, and the Application thereof for Grinding Hyperbolical Glasses hinted at." The illustration accompaning this article is probably not the first picture of a hyperboloid of one sheet, but the article is the first to give a proof that the surface is a doubly ruled surface.

I do not know who first considered the hyperboloid of one sheet. The conic sections were known to the Greeks, but their surfaces of revolution are more problematic. Euclid in his now lost *Surface Loci* may have discussed the hyperboloid of two sheets and this is certainly done by Archimedes in his *On Conoids and Spheroids*, for he computes the volume of a section of the hyperboloid of two sheets [DSB 4, 429]. However, he had only one of the sheets, for until the time of Apollonius the hyperbola was considered to be curve of one branch. In Apollonius's work on *Conics* there is no discussion of surfaces of revolution. It is likely that Kepler discussed the hyperboloid of one sheet in his *Stereometria* (1615), but I am not sure of this.

What Wren published in the 1669 *Philosophical Transactions* is a proof that certain lines on the hyperboloid of one sheet are straight. The geometrical proof that he gives is both easier and more informative than today's proof using analytic geometry[3] so we give it here as presented by Derek T. Whiteside in "Wren the mathematician,"[4] somewhat, we shall stick closely to Wren's original argument.

Ask

When one branch of the hyperbola DBC is revolved about its conjugate axis AO each point D on it generates a circle with center on the axis AO. If AG is the corresponding asymptote then the hyperbola is defined by the relation $DO^2 - GO^2 = BA^2$.

This is the one point of the proof that will seem unfamiliar to the modern reader since we are unaccustomed to such geometric definitions of the conics, so let us pause to get our feet on the ground by converting this equation to cartesian coordinates. If we put $AO = x, DO = y, BA = \alpha$, and λ equal to the slope of the asymptote, then $GO = \lambda \times AO = \lambda x$, and so the definition above has the analytic form $y^2 - \lambda^2 x^2 = \alpha^2$, which is clearly the equation of a hyperbola. It has semi-major axis of length α and semi-minor axis of length α/λ.

Cap. 2

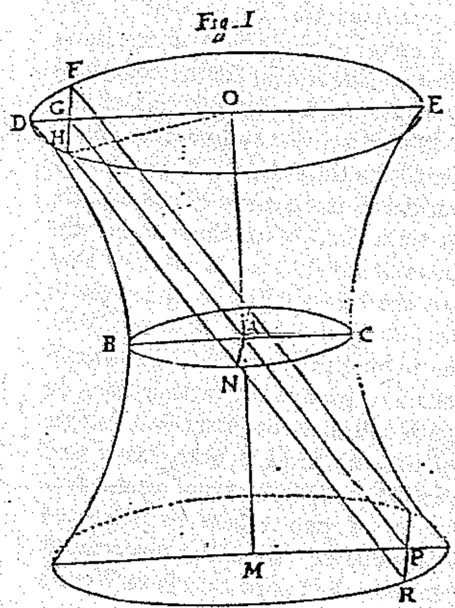
Now consider the intersection HN of the hyperboloid and the plane through the asymptote GA and perpendicular to the plane of the rotated hyperbola. What we wish to show is that HN is a straight line. Since the horizontal sections are circles we have $DO = HO$ and $BA = NA$. Con-

[1] For a careful analysis of the *Discours* and the other mathematical work of Descartes, see J. F. Scott, *The Scientific Work of René Descartes (1596–1650)*.

[2] For a general discussion of Wren's scientific work see J. A. Bennett, *The mathematical science of Christopher Wren*, Cambridge University Press, 1982. Pages 34–38 discuss the hyperboloid.

[3] See Simmons, *Calculus with Analytic Geometry*, p. 579).

[4] *Notes and Records of the Royal Society of London*, 15 (1960), 107–111.



sequently,

$$\begin{aligned} NA^2 &= BA^2 \\ &= DO^2 - GO^2 \\ &= HO^2 - GO^2 \\ &= HG^2 \end{aligned}$$

The second of these lines follows by the definition of the hyperbola, and the last by the Pythagorean Theorem. Thus the "curve" HN is at a constant distance NA from the base plane $DBCE$ of the original hyperbola and so is a straight line lying wholly in the surface of the hyperboloid of one sheet.

We now have an obvious grinding technique: Use an inclined cutting edge HN set obliquely to the axis AO of revolution of the solid to be ground.

There is something here that seems to have escaped notice in our modern texts. If one constructs a string model of the hyperboloid of one sheet then one can "see" the asymptotes by projecting the string that cuts the center of the narrowest part back onto the base plane. For example, the generator HR projects back to the asymptote GP of the generating hyperbola. In fact, there is evidence that it was from such a model that Wren got his idea. He told John Wallis that

> he had once seen a round wicker basket for sale in a shop, among other things, that was woven only from straight pieces of osier[5] lying at oblique angles (and, I believe, crossing each other), the lateral surface of which displayed a cylinder hollowed from without (of that shape which with us is usually given to salt-cellars, or the sheaves of pulleys), he noticed that if one of those osiers led around the axis of a cylinder and preserving that oblique position with respect to the axis would describe that concavo-convex surface, and so cylindroids of that sort could be made on a lathe by means of a straight steel tool held in a position oblique to the axis of the cylinder, the section of which through the axis will be that curved line.[6]

Perhaps you have seen museum models made just this way. If not, see H. Martyn Cundy and A. P. Rollett, *Mathematical Models*, second edition, 1961, pp. 176–178, for information on how to make one for yourself.

In November of that same year 1669 Wren held true to his promise of explaining how to grind lenses in a paper in the *Philosophical Transactions* entitled "A Description of Dr. Christopher Wren's engin, designed for Grinding Hyperbolical Glasses." Here we see two hyperboloids meshing together, one of which is grinding the lens. This also didn't work particularly well, but it does have its use today in another context: hyperbolic gears are used to have one axel drive an oblique one.

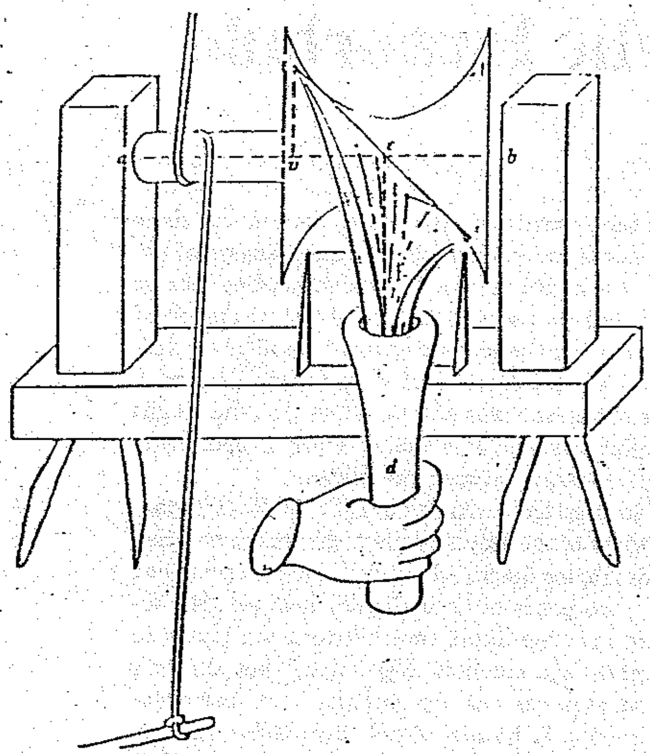
Since the whole problem of grinding lenses was in the air in the seventeenth century it is not surprising that there should be another solution. It was by Newton. In fact he did this while still an undergraduate at Cambridge, although admittedly in the "brisk part of the university" which was influenced by Descartes's scientific view of the world.

The following figure is from Newton's *Mathematical Papers*, 1, 562. It dates from 1666, so if you care about priority, Newton had this discovery before Wren. But then he never published it. It is clear from this figure that Newton understood that the hyperboloid of one sheet is a doubly ruled surface and that one can make one on a lathe by holding the chisel obliquely to the axis.

[The plane and the hyperbolic paraboloid are the only other doubly ruled surfaces, but I do not know who proved this or who coined the phrase.

[5] A species of willow used in basketwork—VFR.

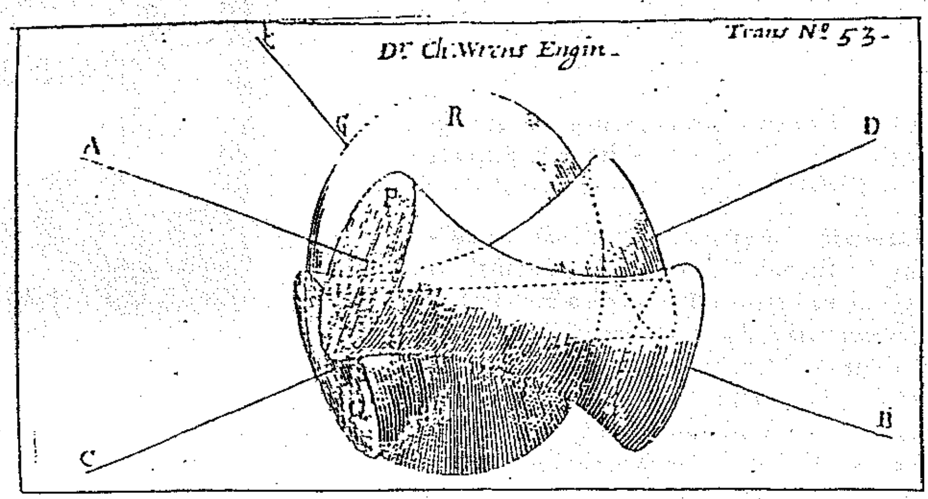
[5] Wallis to René Françoise Sluse, 10 September 1669, *The Corrspondence of Henry Oldenburg*, 6, 237-239.



Neither Newton nor Wren used it. Another of the quadric surfaces, the hyperbolic paraboloid, was discovered by Euler and was first published in 1748 in his *Introductio in analysin infinitorum*, tomus secundus, §122 of the "Appendix de superficiebus."]

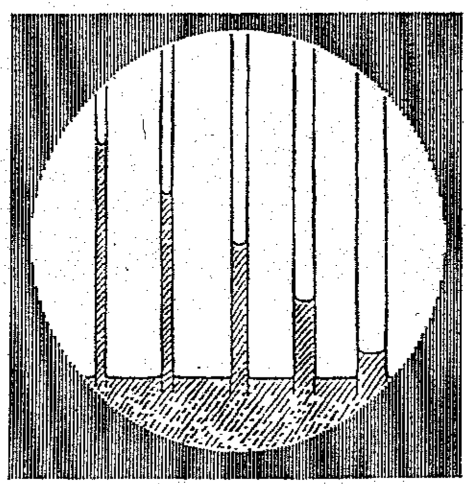
Newton did not succeeed in making a hyperbolic telescope lens, but he did use his ideas in telescope design. That use of the conics is another story.

I wish I had more to say about the hyperboloid, but I do not really know it's subsequent history.



Hauksbee's Hydraulic Hyperbola

Francis Hauksbee (1666–1713) first appeared at a meeting of the Royal Society of London on 15 December 1703, the first meeting presided over by the newly elected president, Isaac Newton. Over the next decade, Hauksbee performed experiments for the Fellows dealing with the air pump, electricity, and capillarity.[1] The rise of fluids in small open tubes had often been noted in the seventeenth-century, but Hauksbee was the first to investigate the phenomenon systematically.[2]



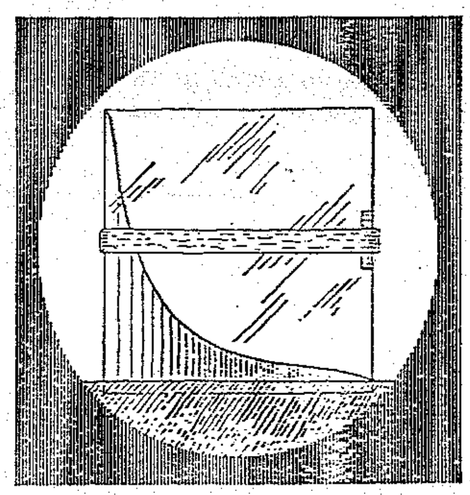
In an early demonstration using the air pump, he placed three small tubes of different diameters in a pan of colored water. Whether or not the air had been evacuated, the water rose in the three tubes, and the smaller the internal diameter of the tube, the higher it rose. In 1708 Hauksbee became convinced that some attractive force was involved and that the phenomena was not peculiar to glass tubes. He performed the experiment again with glass plates and was able to show that the height to which the fluid rose was inversely proportional to the distance between the plates.

In a letter of 25 June 1712 to Hans Sloan, Secretary of the Royal Society, the mathematician Brook Taylor described a similar experiment, but using two panes of glass slightly inclined. He observed that the liquid rose between the plates to resemble "the common *Hyperbola*," but that his "Apparatus was not nice enough," i.e., not accurate enough to be sure of his observation. His letter was read to the Society on 26 June and "Mr. Hauksbee was desired to consider this Letter, and to prepare any Experiments he thinks proper."[3]

[1] "Capillary" derives from the Latin "capillus" meaning "little hair." The effect is definitely noticeable; about 3cm in a tube with a bore of 1mm.

[2] For biographical information on Hauksbee see Henry Guerlac, "Hauksbee, Francis," Dictionary of Scientific Biography, vol. 6, pp. 169–175 and also his "Francis Hauksbee: Expérimentateur au profit de Newton," *Archives internationales d'histore des sciences*, 16 (1963), 124–117.

[3] DSB, 6, 173



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Hauksbee clamped two plates together along one edge but kept them separated at the opposite edge. Then he placed them in water so that, viewed from the top, they formed a V. He carefully measured the cross section of the meniscus (to use the modern word), and confirmed Taylor's conjecture. Further experiments, where the plates were put into the water at an angle, showed "one limb of the hyperbola to be asymptotic to the surface of the water and the other to a line drawn along either of the inclined plates."

The first proof of this result appeared in a work of Musschenbroek in 1762. There is another in Ferguson and Vogel 1926 and a third in Princen 1969.[4] By modern standards, none are correct.[5] Interestingly, as so often happens when a historical topic is considered, one soon is led to research in modern mathematics. This is one of the benefits of an historical approach in the classroom. It would be difficult to mention contemporary research in capillarity were it not for the historical background.

Petrus van Musschenbroek (1692–1761) was a physicist who belonged to a well known family of instrument makers. His father made microscopes for the mathematical physicist Christiaan Huygens and the microscopist Leeuwenhoek. Educated at the University of Leiden, the oldest university in the Netherlands, our Musschenbroek taught natural philosophy and mathematics at Utrecht (1723–1740) and Leiden (1739–??) Musschenbroek was famous for the notes he published about physical experiments, and especially as creator of the Leiden Jar.

This work influenced Newton. See the second edition of his *Optics* (1718), where he inserted several pages into query 23/31. [pp. 315, 366–369].

[4] Musschenbroek, Petrus van, *Introductio ad philosophiam naturalem*, Tom. 1. S.J. et Luchtnams, Leiden, 1762, p. 376. Ferguson, A., and Vogel, I., "On the 'hyperbolic' method for the measurement of surface tensions," Phys. Soc. Proc. 38(1926), pp. 193–203. Princen, H. M., "Capillary phenomena in assemblies of parallel cylinders. II Capillary rise in systems with more than two cylinders," *Journal of Colloid Interface Science*, 30 (1969), p. 359–371.

[5] Finn, Robert, *Equilibrium Capillary Surfaces*, Springer, 1986, note 4, p. 131.

A Proof

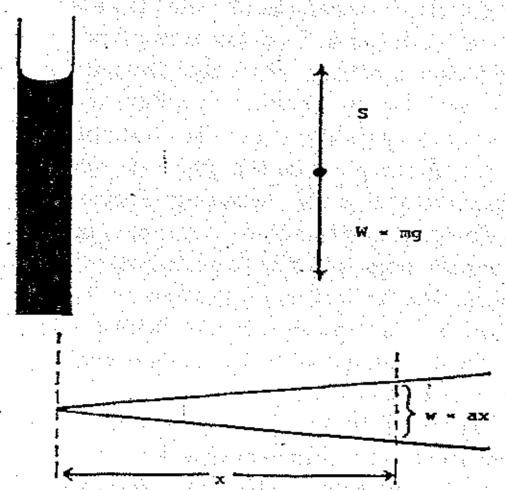
Suppose that the distance between the two glass plates, clamped together at one edge, are placed in a fluid so as to form a 'V' when viewed from the top. The fluid will rise between them until the surface tension is balanced with the weight of water below the surface. Consider an infinitesimally thin vertical cross section that is distance x from the point of the wedge. At this point, the plates are distance w apart, which is proportional to x, so $w = ax$. The net surface tension acts directly upward and this force S is balanced by the weight of water below, which is proportional to the cross sectional area A_x of the column of fluid below. Thus $S = kA_x$. Now the area A_x is equal to the height of the column h times its width w. Putting all of this together, we have,

$$S = kA_x = jwy = kaxy.$$

Solving for y we have

$$y = \frac{S}{ka}\frac{1}{x},$$

which is the equation of a hyperbola.[6]



[6] John W.Dawson Jr., "Displaying the conics. Three Alternatives to computer graphics," *Primus*, 1 (1991), 87–94.

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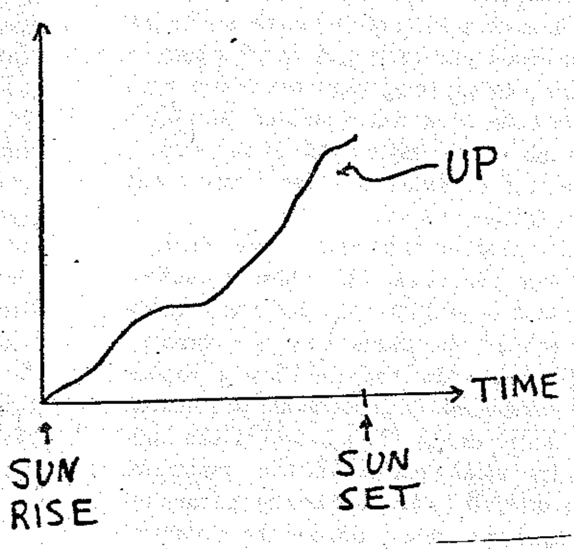
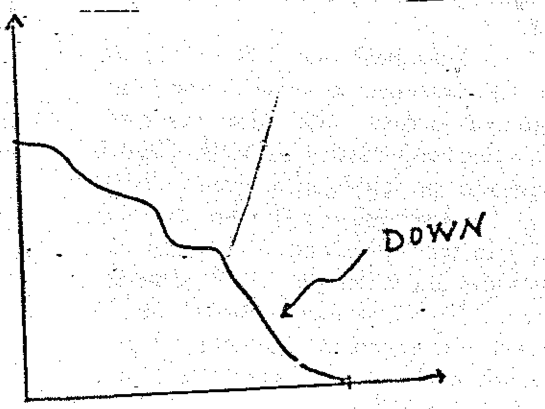
Bolzano's Intermediate Value Theorem

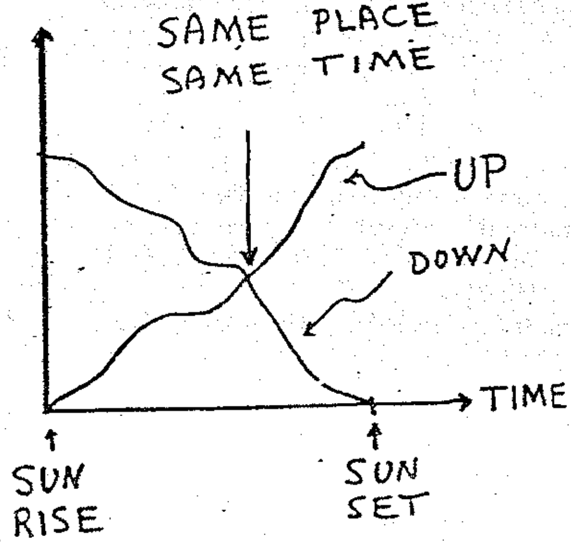
When introducing the intermediate value theorem I begin with the following story that ends with a question (taking care to pose it at the end of a class, so the students have an evening to think about it):

> One day a monk leaves at sunrise to climb up a mountain. He walks at a leisurely pace, sometimes stopping to enjoy the view, even retracing his path to look again at a pretty flower. He arrives at the summit at sundown, spends the night meditating, and starts home down the same path the next day at sunrise, arriving home at sunset. The question is this: Was there a time of day when he was exactly at the same point on the trail on the two days?





The next class begins by bringing out the hidden assumptions of this ill posed problem (doing this is one of the strengths of the problem). I do this by getting the students to talk about the problem, and, most often, why they cannot solve it. The monk travels along the same path on both days and his position is determined by the distance from the bottom of the path. Position is, of course, a continuous function of time. If we plot the path up the mountain in a time-distance coordinate system, then the curve goes from (sunrise, bottom) to (sunset, top). Flat regions on the graph are rest times, and dips arise from, say, retracing his steps to look at a flower. The path down the mountain is a curve from the point (sunrise, top) to (sunset, bottom). When the two paths are plotted on the same axes, it is obvious that the curves intersect—this is a point where the monk is at the same point at the same time on the two days.



There is an insightful solution to the problem that is equivalent to this graphical one, but there is no need to draw any picture. One monk and two days does not make the solution as transparent as it could be so we use Polyá's technique of looking at a similar (and in this case, equivalent) problem. Suppose there are two monks and they both start at sunrise, one at the bottom of the mountain, the other at the top. Every student will see that they must meet somewhere along the path—at the

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Then, I point out that the theorem we used here, which we call the Intermediate Value Theorem, is due to Bernhard Bolzano (1781–1848), who was, in fact, a monk. His mother was a pious woman, his father an Italian immigrant who earned a modest living as an art dealer. His father was widely read and felt responsible for his fellow men. This was not just theory; he took an active part in founding an orphanage in Prague. His son, Bernhard, studied philosophy, physics, and mathematics at the University of Prague in his native city. It was this grounding in philosophy and logic that convinced him of the necessity of formulating clear concepts and of using sound reason to deduce theorems from irreducible first principles. His interest in mathematics was stimulated by B. Kästner's *Anfangsgründe der Mathematik*, a book where the author took care to prove statements which were commonly regarded as evident in order to make clear the assumptions on which they depended.[2]

After graduating in 1800, Bolzano entered the theological faculty at the University of Prague and was ordained a Roman Catholic priest in 1804. In 1805, Emperor Franz I of Austria, of which Bohemia was then a part, established a chair of philosophy in each university. His reasons were mainly political, as he feared the spread of the ideas which had fomented the French Revolution and which were widespread in Bohemia. In 1805 Bolzano was appointed to the new chair of Philosophy of Religion at the University of Prague. His unorthodox religious and political ideas made him quite unsuitable for this position. However his lectures and sermons were exceptionally popular among the students in that he advocated human rights and utopian socialism. These views, as well as his unorthodox religious views, led to his dismissal on 24 December 1819. He was forbidden to publish and was put under police supervision, but he refused to recant. The remainder of his life was spent working on philosophy and mathematics while living with friends.

1781-1848

BERNARD BOLZANO

Engraving by [illegible] after a drawing by Kriehuber, published in *Starý Světozor*, [illegible] (Picture by courtesy of the Museum of Czech Literature.)

In Prague he was isolated from the center of the mathematical world in Paris. The fact that he held a university post for only a few years also contributed to the fact that his mathematical ideas received little recognition until Herman Hankel and Otto Stolz called attention to them in 1871 and 1881 respectively.[3] Then they rapidly became well known.

But we are way ahead of the story. The work that interests us is Bolzano's now famous "Rein analytischer Beweis" of 1817, which has the full title "Purely analytic proof of the theorem that

[1] See James C. Frauenthal and Thomas L. Saaty, "Foresight—Insight—Hindsight," pp. 1–22 (especially pp. 3–4) in *Discrete and System Models* edited by William F. Lucas, Fred S. Roberts, and Robert M. Thrall, Springer, 1976. This is volume 3 in the series *Modules in Applied Mathematics*.

[2] For additional information about Bolzano, see B. Van Rootselaar, "Bolzano, Bernard," *Dictionary of Scientific Biography*, vol. 2, pp. 273–279.

[3] Herman Hankel (1839–1873), "Grenze," *Allg. Encl. Wiss. Kunst* (1871), sect. 1, part 90, pp. 185-211, Leipzig. Otto Stolz (1842–1905) "B. Bolzano's Bedeutung in der Geschichte der Infinitesimalrechnung," *Mathematische Annalen*, 18 (1881), pp. 255–279.

between any two values which give results of opposite sign there lies at least one real root of the equation."[4] This is the theorem which we now call the intermediate value theorem. In this paper he also gives the definition of continuity that we still use today.

Bolzano makes the claim that his theorem "clearly rests on the more general truth that, if two continuous functions of x, fx and ϕx, have the property that for $x = \alpha$, $f\alpha < \phi\alpha$, and for $x = \beta$, $f\beta > \phi\beta$, there must always be some value of x lying between α and β for which $fx = \phi x$."[5] Although Bolzano doesn't draw a picture we should draw one for our students, because then the solution to the monk's problem becomes transparent.

Bolzano made a distinction in this paper which we should heed as teachers. He is not interested in giving a mere 'confirmation' of the theorem, but wished to give a 'justification' of it. He points out that the theorem is perfectly obvious and does not need confirmation. He provides a justification or proof of the theorem because he is interested in the foundations of analysis, a pursuit in which he was far ahead of his contemporaries. If we don't want to be far ahead of our students then we should dispense with a justification of the intermediate value theorem and concentrate on it's confirmation. This is a case where history tells us what not to prove in the classroom.

The same mathematical results are in Cauchy's *Cours d'Analyse* of 1821. This coincidence has prompted Grattan-Guinness, in his doctoral dissertation, to look for an explanation of these similarities.[6] He concluded that Cauchy stole this definition from Bolzano, citing similarities in the work, the fact that the journal which carried Bolzano's paper began to appear in the Paris libraries with the very issue in which the "Rein Beweis" appeared, the fact that Cauchy read German, was careless in citing his sources, and was a rather nasty man. This charge is now generally regarded as unfounded [Freudenthal, Sinaceur, as well as much work done recently on Bolzano that does not specifically deal with this *ex post facto* priority dispute.] Grabiner concludes that while Grattan-Guinness gave the wrong answer, he did ask the right question.[7]

The Intermediate Value Theorem is a theoretical tool in our calculus classes today (I am still working on how it attained this role in the nineteenth-centruy). There are very few applications. The type of problems that is most prevalent in our texts, that of finding the point c where the function assumes the intermediate value, is a bogus problem, and should be banned. If we knew how to find c we would not need to use the IVT. There are only a few problems that I know of that make essential use of the IVT:

(1) From my classroom in Bowling Green, I ask if there is a direction that I can point such that the temperature at the boundary of the State of Ohio is the same in that direction and in the opposite direction? I hold my arms out at my sides and point in opposite directions, and then swing around as I ask the question. Define a function as the difference in temperatures at the state boundary between where my right hand points and where my left points. Fix a direction. If the function is zero there, we are done. If not, turn 180°. For this direction the function will have the opposite value. By the IVT, the function must be zero for some intermediate direction. I don't know if this argument will work in your state, but if it fails, it will be instructive to see why.

(2) Must it happen at some time in your life that your height in feet equals your age in years?

(3) Stretch a rubber band. Is there some point that does not move?

These are the only interesting problems that I know which use the Intermediate Value Theorem. Part of their charm is that each of them has hidden assumptions and conditions which need to be brought out. If you have other such problems, I would be happy to hear of them. I would also like to know about the history of these problems, but I haven't a clue.

---

[4] "Rein analytischer Beweis des Lehrsatzes dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleighung liege," *Abh. Gesell. Wiss. Prague*, (3), vol. 5, (1814–1817), pp. 1–60. English translation by S. B. Russ in *Historia Mathematica*, vol. 7(1980), 156-185.

[5] Bolzano, op. cit., p. 166]

[6] "Bolzano, Cauchy and the 'new analysis' of the early nineteenth century," *Archiv for the History of Exact Sciences*, vol. 6 (1970), pp. 372–400. *The Development of the Foundation of Mathematical Analysis from Euler to Riemann*, MIT press, 1970.

[7] Judith V. Grabiner, 'Cauchy and Bolzano. Tradition and transformation in the history of mathematics," pp. 105–124 in *Transformation and Tradition in the Sciences* (1984), edited by Everett Mendelsohn, Cambridge University Press.

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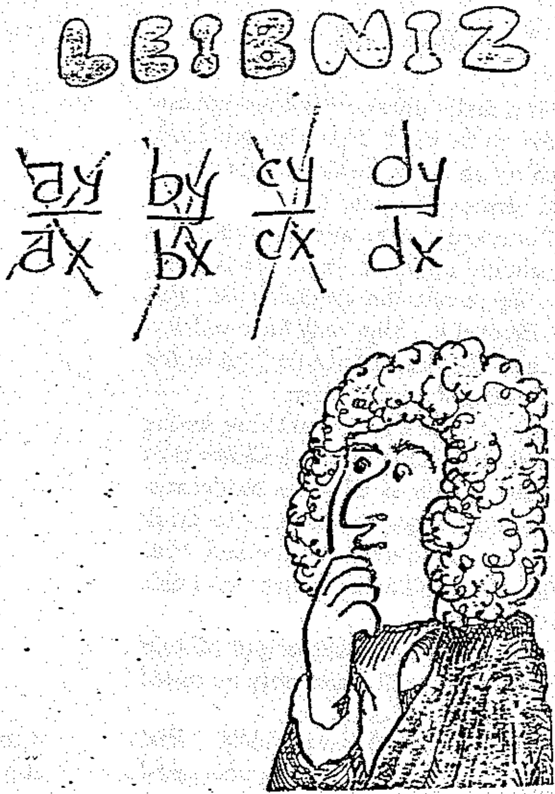
The Product Rule



One thing we should not neglect to tell our students, for it is especially encouraging to the beginner, is that Leibniz had considerable difficulty discovering the correct form of the product rule. It is exciting to follow his struggle in his manuscripts during 1675–1676 when he was in the process of inventing the calculus. Indeed, these are some of the most precious documents in the whole history of mathematics. Among other things, they provide a wonderful example of how mathematics is done.

In a manuscript dated 11 November 1675, Leibniz introduced the differential notation dx. He thought of a variable as taking on a sequence of values and he was considering *differences* of these.[1] This is why he chose the letter d — it stands for 'difference,' not for 'differential' or 'derivative' as we are inclined to believe today.[2] My colleague Vic Norton conceives of Leibniz's choice of notation in a more humorous way (see the figure).

In this same manuscript of 11 November, 1675, Leibniz wrote "Let us now examine whether $dx\,dy$ is the same thing as $d\overline{xy}$, and whether dx/dy is the same thing as $d\frac{x}{y}$." Here he has used the overbar where we would use parentheses. In order to test this conjecture he considered an example: He took $x = cz + d$ and $y = z^2 + bz$ and then correctly computed $dx\,dy$. Then in the rush of discovery he added, "But you get the same thing if you work out $d\overline{xy}$ in a straightforward manner." But he neglected to do it! Consequently, we have the makings of a good problem to give our students—continue the example and draw the conclusion that Leibniz should have drawn.

Later in the same manuscript, after noting the absurdity of $\int \overline{d\nu\, d\psi} = \int d\nu \int d\psi$, he writes[3]

> Hence it appears that it is incorrect to say that $d\nu\, d\psi$ is the same thing as $d\overline{\nu\psi}$, or that $\frac{d\nu}{d\psi} = d\frac{\nu}{\psi}$; although just above I stated that this was the case, and it appeared to be proved. This is a difficult point. But now I see how this is to be settled.

It is not clear what he meant by "appeared to be proved" but he settled the difficulty by counterexample, taking $\nu = \psi = x$, augmenting x by dx, and then computing:

$$d(x^2) = (x + dx)^2 - x^2 = 2x dx$$

without even a mention of what happened to the $(dx)^2$. Next he wrote

$$dx\, dx = (x + dx - x)(x + dx - x) = (dx)^2.$$

---

[1] Henk J. M. Bos, "The fundamental concepts of the Leibnizian calculus," *Studia Leibnitiana*, Sonderheft 14 (1986), 103-118; reprinted in his *Lectures in the History of Mathematics* (1993), AMS, pp. 83-99.

[2] The concept of derivative came much later. See Judith V. Grabiner, "The changing concept of change: The derivative from Fermat to Weierstrass," *Mathematics Magazine*, 56 (1983), pp. 195–203; reprinted in Frank Swetz, *From Five Fingers to Infinity* (1994), pp. 607-619.

[3] Leibniz had invented the integral sign thirteen days previously, on October 29, 1675.

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In this paragraph we have changed the notation, while previously Leibniz's notation has been carefully preserved.

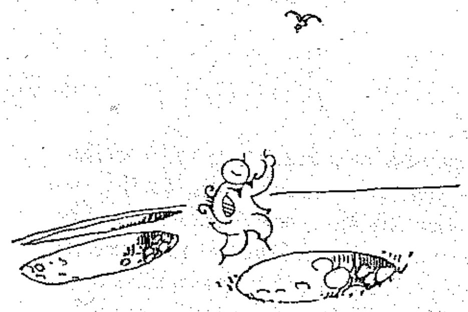
Ten days later, on 21 November 1675, Leibniz has the product rule, but stated in the form

$$\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}.$$

He notes "this is a really noteworthy theorem and a general one for all curves." Then he cryptically adds "But nothing new can be deduced from it, because we had already obtained it."

By way of encouragement and motivation to the student we should point out that it took Leibniz ten days to figure out the product rule. But then he had to discover it. They only have to learn to use it. But then they had better do that in ten days—or risk flunking the next exam.

I also point out that there is nothing wrong with making mistakes. This example shows that one of the greatest minds of all times made mistakes. What is wrong is not to continue to think about what you have done until you are sure that everything is OK. The following 'grook' says this in a more positive light.[4]



THE ROAD TO WISDOM

The road to wisdom? —Well, it's plain
and simple to express:
Err
and err
and err again
but less
and less
and less.

It is not until Leibniz's manuscripts of July 1677 that we find what might reasonably be called a proof of the product rule, but we shall not quote it here[5] because the following proof, given in a letter which Leibniz wrote Wallis on March 30, 1699, while essentially the same, is somewhat clearer:

> It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x+dx$, then dx is rejected. But it is different

[4] Piet Hein, *Grooks*, MIT press, 1966. Piet Hein (b. 1905) is a Danish engineer, poet, and intellectual jack-of-all-trades. As a friend of many mathematicians, he has applied his skills to both architecture and games. He invented the "super-ellipse" $|\frac{x}{a}|^p + |\frac{y}{b}|^p = 1$ as the shape of a traffic circle for a rectangular "square" in Stockholm City Center. In 1942 Hein invented the game of Hex (the American mathematician and Nobel laureate in Economics John Nash proved that the first player always wins). He also invented the SOMA Cubes which perplexed thousands in the 1960s (the name is a registered trademark of Parker Brothers). His "Grooks" are delightful, short, aphoristic poems, each accompanied by one of his drawings. Originally they were a kind of underground language—beyond the German comprehension, and way beyond their sensibilities—used by the Danish Resistance to the Nazi's in World War II. More recent Grooks in English apply his wit and wisdom to the human condition. Several are ideal for use in the classroom. Hein is truly a great Dane. For a picture of Hein see p. 328 of Anatole Beck, Michael N. Bleicher, and Donald W. Crowe, *Excursions into Mathematics*, New York: Worth, 1969.

[5] All of the above quotations are taken from "The manuscripts of Leibniz on his discovery of the differential calculus," *The Monist*, **26**(1916), 577–629, **27**(1917), 238–294 and 411–454; esp. pp. 279–281, 286 and 439. This paper is a translation, with highly unreliable commentary by J. M. Child, of papers by C. I. (or K. J.) Gerhardt, who discovered the papers in the Royal Library of Hanover in the mid-nineteenth century. They are also in J. M. Child's *The Early Mathematical Manuscripts of Leibniz. Translated from the Latin Texts Published by Carl Immanuel Gerhardt with Critical and Historical Notes.* Chicago, London: Open Court, 1920, iv + 238 pp. See pages 100–102, 107 and 143 for the passages quoted.

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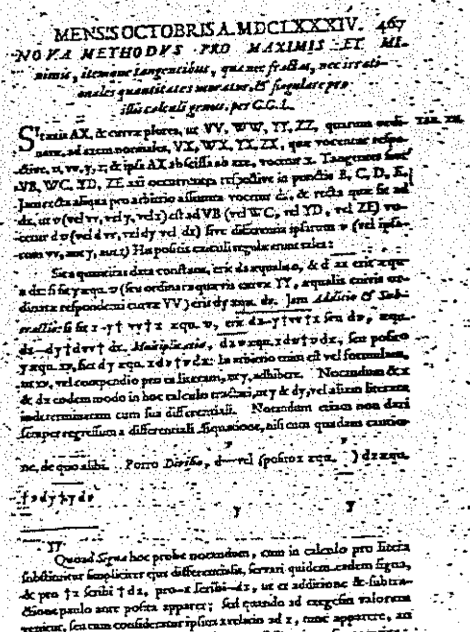
if we seek the difference between $x+dx$ and x, for then the finite quantities disappear. Similarly we cannot have $x\,dx$ and $dx\,dy$ standing together. Hence if we are to differentiate xy we write:

$$(x+dx)(y+dy) - xy = x\,dy + y\,dx + dx\,dy$$

But here $dx\,dy$ is to be rejected as incomparably less than $x\,dy + y\,dx$. Thus in any particular case the error is less than any finite quantity.[6]

The product rule first appeared in print in 1684 in Leibniz's first paper on the differential calculus: "A new method for maxima and minima as well as tangents which is neither impeded by fractional nor irrational quantities, and a remarkable type of calculus for them." See if you can locate the statement of the product rule on the first page of this famous paper of Leibniz: [7]



It next occurred in Newton's *Philosophiae naturalis principia mathematica*, book 2, lemma 2, although Newton had known the result since 1665.[8] Newton's proof of the result was harshly criticized by George Berkeley in his *Analyst* of 1734. "The *Analyst* is more than a Christian apologetic; it is a work on mathematics for mathematicians,"[9] and profoundly influenced the development of the foundations of the calculus. Here is what Berkeley says about the product rule:

> I proceed to consider the principles of this new analysis ... wherein if it shall appear that your capital points, upon which the rest are supposed to depend, include error and false reasoning; it will then follow that you, who are at a loss to conduct your selves, cannot with any decency set up for guides to other men. The main point in the method of fluxions is to obtain the fluxion or momentum of the rectangle or product of two indeterminate quantities. ... Now, this fundamental point one would think should be very clearly made out, ... But let the reader judge. This is given as demonstration [by Newton]. Suppose the product or rectangle AB increased by continual motion: and that the momentaneous increments of the sides A and B are a and b. When the sides A and B were deficient, or lesser by one half of their moments, the rectangle was $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$ i.e., $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$. And as soon as the sides A and B are increased by the other two halves of their moments, the rectangle becomes $\overline{A + \frac{1}{2}a} \times \overline{B + \frac{1}{2}b}$ or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From the latter rectangle subduct the former, and the remaining difference will be $aB + bA$. Therefore the increment of the rectangle generated by the entire increments a an b is $aB + bA$. *Q.E.D.* But it is plain that the direct and true method to obtain the moment or increment of the rectangle AB, is to take the sides as increased by their whole increments, and so multiply them together, $A + a$ by $B + b$, the product whereof

[6] *Leibnitzens Mathematische Schriften*, IV, 63, edited by Gerhardt. Translation from D. E. Smith, *History of Mathematics*, vol. 2, p. 696–697.

[7] For an English translation of this paper, see D. J. Struik, *A Source Book in Mathematics, 1200–1800*, pp. 271–280.

[8] *The Mathematical Papers of Isaac Newton. Volume 1. 1664-1666*, edited by D. T. Whiteside, Cambridge University Press, 1967, pp. 344, 383, and 402.

[9] So say A. A. Luce and T. E. Jessop on p. 58 of the editor's introduction to *The Works of George Berkeley, Bishop of Cloyne*, London: Thomas Nelson and Sons, 1951, vol. 4.

$AB + aB + bA + ab$ is the augmented rectangle; whence, if we subduct AB the remainder $aB + bA + ab$ will be the true increment of the rectangle, exceeding that which was obtained by the former illegitimate and indirect method by the quantity ab. And this holds universally be the quantities a and b what they will, big or little, finite or infinitesimal, increments, moments, or velocities. Nor will it avail to say that ab is a quantity exceedingly small: since we are told that *in rebus mathematicis errores quam minimi non sunt contemnendi.* [The most minute errors are not in mathematical matters to be scorned.][10]

Berkeley's criticism here is right on the mark. Mathematicians were unable to give a better proof of the product rule until Cauchy introduced the definition of the derivative using limits in his *Cours d'Analyse* of 1821.

Exercises

1. "A function $u(x)$ being given, it is required to determine a formula giving all the functions $v(x)$ for which the derivative of the product u and v is equal to the product of their derivatives."[11]
2. Verify this alternate proof of the product rule: First get $\frac{d}{dx}[f^2(x)]$ from the definition (which is an interesting exercise anyway) and then use the identity $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$ to finish the proof.[12]
3. Maria Agnesi in provided an easy approach to the quotient rule in her *Instituzione Analitiche* of 1748: If $h = f/g$, then $hg = f$. Now apply the product rule, substitute f/g for h and then solve for h'.


ACTA
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ANNO MDCLXXXIV
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SERENISSIMO FRATRUM PARI,
DN. JOHANNI
GEORGIO IV,
Electoratus Saxonici Hæredi,
&
DN. FRIDERICO
AUGUSTO,
Ducibus Saxoniæ &c &c. &c.
PRINCIPIBUS JUVENTUTIS
dicata,
Cum S. Cæsareæ Majestatis & Potentissimi Elect. Saxon. Privilegiis.
LIPSIÆ,
Prostant apud J. GROSSIUM & J. F. GLEDITSCHIUM.
Typis CHRISTOPHORI GÜNTHERI.
Anno MDCLXXXIV.

[10] Berkeley's footnote is *"Introd. ad Quadraturam Curvarum."* This refers to Newton's "Tractatus de quadratura curvarum," the second appendix to his*Optics* (1704). The line Berkeley quotes in Latin is from page 167, but he permuted the word order. See *The Mathematical Papers of Isaac Newton*, vol. VIII, pp. 124–5. The long passage quoted above is from Berkeley's *The Analsyt* (1737) which has been reprinted in Luce and Jessop, op. cit., vol. 4, pp. 53–102; see §9, pp. 69–70.

[11] This problem is from the 0^{th} Putnam exam, which was held May 19 and 20, 1933. See David C. Arney, "Army beats Harvard in football and mathematics," *Math Horizons*, September 1994, pp. 14–17. Of course Army won; the score was 112 to 98.

[12] Russell Euler, "A note on differentiation," *The College Mathematics Journal*, 17(1986), 166–167.

Why the sine has a simple derivative

S[r] Isaac Newton,
speaking of M[r] Cotes, said
"If He had lived
we might have known something."[1]

Many advocate the use of writing in the teaching of mathematics and there is much to recommend it. David G. Hartz, now of the College of Saint Benedict in Minnesota, has long had the habit of requiring his students to write a short summary of the most important mathematical points that had arisen in class that week and their reaction to them. He wants them to explain, for another student in the class, what they did the previous week, and to do it in their own words, keeping formulas to a minimum. The purpose of this assignment was to encourage students to review what they had done the previous week and to reflect about it. Here is what one student wrote:

> The derivatives of the trigonometric functions are rather amazing when one thinks about it. Of all the possible outcomes, $D_x \sin x = \cos x$. Simply $\cos x$; *not*
>
> $$\frac{1}{542}\cos x\left(\frac{1}{\pi}\right)\cdot 2x.$$
>
> But simply $\cos x$. Is it just *luck* on the part of the mathematicians who derived trig and calculus? I assume trig was developed before calculus, why or how could the solution prove to be so simple? Luck.[2]
>
> A. Student
> Fl. 1988

Now it would be easy to dismiss this rather exuberant student as one in a creative writing class. But what we realize from these comments is that the student has not come to grips with at least one important mathematical idea, thinking instead that the formula $D_x \sin x = \cos x$ is a matter of luck, "just *luck*." Now we mathematicians know that this is not the case, but we must explain it to our students, or, better yet, provide exercises that lead them to discover this for themselves.

We measure angles in radians, but this student does not understand why we do. One way to rectify this view is to ask for $D_x \sin x^\circ$, where x is measured in degrees. Since 2π radians = 360 degrees we have $1^\circ = 2\pi/360$. Consequently,

$$\begin{aligned} D_x \sin x^\circ &= D_x \sin\left(\frac{2\pi}{360}x\right) \\ &= \frac{2\pi}{360}\cos\left(\frac{2\pi}{360}x\right) \\ &= \frac{2\pi}{360}\cos x^\circ. \end{aligned}$$

No one wants to write the factor $2\pi/360$ each time they take the derivative of $\sin x$. This is the mathematical reason why radians are used.

What I like best about the above quotation is that it cries out for some historical comments. The student reasonably assumes that trigonometry was developed before calculus since it is such an integral part of our calculus courses. But it wasn't that simple. Technically the student is correct, but not in spirit. Trigonometry was developed by the Greeks, but there were no trigonometric *functions* until Euler introduced them in 1739 in order to solve differential equations, specifically linear differential equations with constant coefficients.[3]

A Discovery of Cotes

In his *Harmonia mensurarum* of 1722, Roger Cotes (1682–1716) states and proves the following lemma:

[1] So wrote Robert Smith, Cotes's cousin and successor as Plumian Professor of Astronomy at Trinity College, Cambridge, in his copy of Cotes's *Harmonia mensuarum*. See Ronald Gowing, *Roger Cotes. Natural Philosopher*, Cambridge University Press, 1983, pp. 141–142, for a nice illustration of the original.

[2] David G. Hartz, "Writing abstracts as a means of review," pp. 101–103 in *Using Writing to Teach Mathematics*, MAA Notes Number 16, edited by Andrew Sterrett.

[3] Victor J. Katz, "The calculus of the trigonometric functions," *Historia Mathematica*, 14(1987), 311-324. Much of the historical material in this note derives from this paper.

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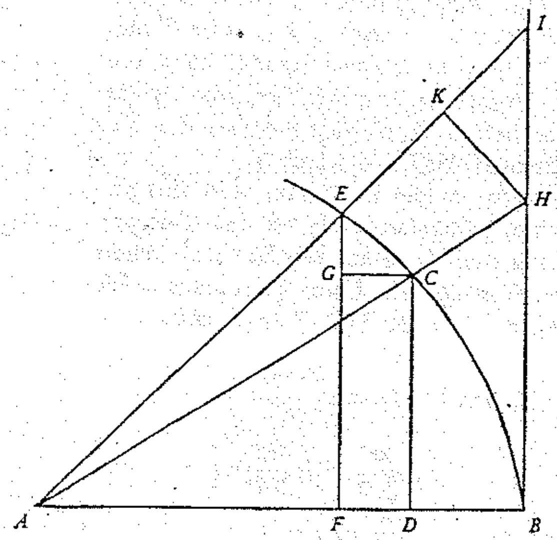
The small variation of any arc of a circle is to the small variation of the sine of that arc, as the radius to the sine of the complement.

Proof: With centre A, radius AB, arc BC is drawn; CG is parallel to the radius AB; CE is the variation of arc BC; EG is the variation in the sine of that arc. I say $CE : CG :: AC : AD$, when the variations are very small.[4]



[4] Quoted from Gowing, op. cit., p. 93.

Reformulating this in modern notation, the "small variation of any arc of a circle" becomes $d\theta$ while "the small variation of the sine of that arc" is $d(\sin\theta)$. Hence the formula is

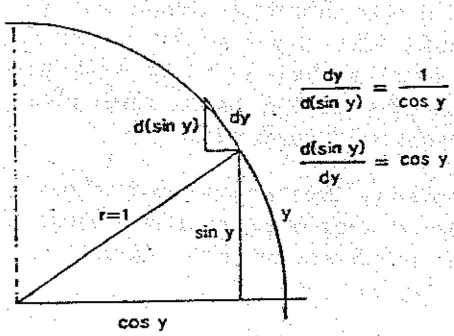
$$\frac{d(r\theta)}{d(\sin\theta)} = \frac{r}{\cos\theta}.$$

In 1748, Euler simplified this by using the unit circle. Thus:

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta,$$

where r is the radius of the circle under consideration.

To see why this is obviously true, note that if the arc BC has measure θ and EC is a small increment of that arc, then it is $d\theta$. The line segment DC is $\sin\theta$ and so GE is $d(\sin\theta)$. In the triangle $\triangle ADC$, radius AC has, by convention, length 1 and side AD is $\cos\theta$. Since $\triangle ADC$ is similar to $\triangle EGC$, we have the desired result.



Those who object to the lack of rigor in this differential argument should realize that theorems have to be discovered before they are proved. What we are really after here is an intuitive explanation of why the derivative of the sine is the cosine.

The Word 'radian'

Curiously, the word "radian" entered mathematics rather late — not until 1871, and then only in the private papers of James Thomson, brother of Lord Kelvin (nee John Thomson). More interestingly, it first occurred in an examination paper, dated June 5, 1873, at Queen's College, Belfast. The next year John Muir adopted the term after consultation with Thomson.[5]

[5] Florian Cajori, *A History of Mathematics*, second edition, 1924, p. 484. He cites *Nature*, vol, 83, pp. 156, 217, 459, 460.

It was not a matter of luck. The choice of radians as a measure for angles was the result of careful thought.

Exercises

1. "See how many first year mathematics graduate students understand that the derivative of $\sin\theta$ is equal to $\cos\theta$ only if θ is measured in radians. (Ask them to plot $\sin\theta$ vs. θ from 0° to 90° on a graph, and then measure the slope at 30°.)"[6]
2. In the diagram, show that $KH = \sec\theta \cdot d\theta$, $KI = d(\sec\theta)$ and $HI = d(\tan\theta)$. Then, using similar triangles, find the formulas for the derivatives of $\tan\theta$ and $\sec\theta$.

Bust of Roger Cotes, in the Wren Library at Trinity College.

[6] Clifford E. Swartz, "What do physics teachers want?," *The UMAP Journal*, vol. 1, no. 1 (1980), pp. 115–125, esp. p. 116. This physics professor at Stony Brook reports that students are coming to college with as much mathematics as ever, some even with calculus, but that "They just can't use mathematics." He outlines the rather minimal amount of mathematics that physics students need to know — and *use* with some sophistication.

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L'Hospital's Rule

Guillaume-François-Antoine de L'Hospital, Marquis de Sainte-Mesme, Comte d'Entremont was born in Paris in 1661 and died there in 1704. At the age of fifteen, L'Hospital surprised his elders with his mathematical talent when he solved one of Pascal's problems on the cycloid. As a wealthy French nobleman, it was natural that L'Hospital would enter the cavalry. He rose to the rank of captain, but then took advantage of his wealth and social position and resigned to devote the rest of his life to the pursuit of mathematics. Lest you think this terribly altruistic, I should point out that his exceptional nearsightedness was a severe handicap for a cavalry officer. Little is known about l'Hospital as an individual, but "According to the testimony of his contemporaries, L'Hospital possessed a very attractive personality [being], among other things, modest and generous, two qualities which were not widespread among mathematicians of his time."[1]

L'Hospital retired to Paris where, by 1689, he became a member of the scientific circle of Nicolas Malebranche (1638–1715), a diligent and enthusiastic amateur mathematician. Others mathematicians involved were Carré, Reyneau, and Varignon. There is no doubt that L'Hospital was the most capable mathematician in this group and was probably the best French mathematician of his day.

When Johann Bernoulli (1667–1748) visited Paris in the fall of 1691 he quickly impressed the group with his knowledge of the new calculus of Leibniz. Especially impressive was the formula he possessed for finding the curvature of curves. He neglected to point out that the formula was due to his older brother Jakob. The two Bernoulli boys had studied the two calculus papers of Leibniz that had appeared in *Acta eruditorum*, his 1684 paper on the differential calculus and the 1686 paper on the integral calculus. These papers were extremely densely written, full of misprints, and deliberately obscure at critical points (to forestall criticism of his work, Leibniz made his differentials finite quantities rather than infinitesimals). Jakob Bernoulli had written to Leibniz for clarification, but Leibniz was away when the letter arrived. By the time a response was received the two brothers had mastered the new calculus on their own. This accomplishment is a true indication of their genius. The mathematicians in France, led by L'Hospital, had made some inroads into the differential calculus paper, but were completely baffled by that on the integral calculus. Thus it is not surprising that the 24 year old Johann Bernoulli

Guillaume François Marquis de l'Hôpital

1661–1704

[1] Abraham Robinson (1918–1974), "L'Hospital," *Dictionary of Scientific Biography*, vol. 8 (1973), pp. 304–305, esp. p. 305. Also see Robinson's paper "Concerning the history of the calculus," in his *Non-Standard Analysis*, North-Holland, 1966, pp. 260–282, which discusses the foundations of L'Hospital's work, especially his use of infinitesimals.

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made a big splash when he arrived in Paris in 1691.

It was not long before the Bernoulli boys progressed from fellow students of the calculus to rival researchers. No full study of the interactions and clashes of the Bernoulli brothers is available, but their rivalry soon spilled over into print. Let me give just one quotation to show the personality of the younger brother, Johann. The first independent work of Johann Bernoulli was to solve the catenary problem, i.e., to determine the shape of an elastic cable that is suspended between two points under its own weight.[2] He was extremely proud that he had solved this problem and that his brother Jakob, who had posed it, had not. Writing to Pierre Remond de Montmort (1678–1719) years later, on 29 September 1718, he boasted:

> The efforts of my brother were without success; for my part, I was more fortunate, for I found the skill (I say it without boasting, why should I conceal the truth?) to solve it in full and to reduce it to the rectification of the parabola. It is true that it cost me study that robbed me of rest for an entire night. It was much for those days and for the slight age and practice I then had, but the next morning, filled with joy, I ran to my brother, who was still struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more to try to prove the identity of the catenary with the parabola, since it is entirely false. The parabola indeed serves in the construction of the catenary, but the two curves are so different that one is algebraic, the other is transcendental.[3]

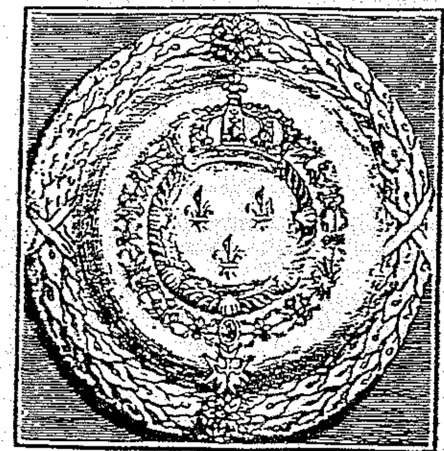
In 1691, when Johann Bernoulli (1667–1748) met L'Hospital, who was at the time particularly interested in learning about the new calculus of Leibniz, he again took advantage of his wealth and hired Bernoulli to teach him. This was a significant event for both of them. Bernoulli, who

ANALYSE

DES

INFINIMENT PETITS,

Pour l'intelligence des lignes courbes.



A PARIS,

DE L'IMPRIMERIE ROYALE.

M. DC. XCVI.

was newly married and unemployed, spent four months tutoring L'Hospital in the calculus, first in Paris and later at L'Hosptial's country estate of Ouques. In return, L'Hospital, with assistance from his friend Christiaan Huygens, obtained a university position for Bernoulli at Groningen in the Netherlands in 1695.

Today L'Hospital's name is only associated with the rule for evaluating limits of indeterminate forms, but in his day and for several generations thereafter, L'Hospital's fame rested on his book *Analyse des infiniment petits, pour l'intelligence des lignes courbes*, which was published anonymously in Paris in 1696. The title of this book can be very loosely translated as "Analysis using Infinitesimals for the Study of Curved Lines." Note that it refers to the study of curves, not functions; the function concept had not yet been introduced into mathematics (Euler did it in the next century). This was the first textbook on the differential calculus, as so deserves our special attention.

Toward the end of the seventeenth-century

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[2] *Acta eruditorum*, 1691.

[3] *Der Briefwechsel von Johann Bernoulli, (1667–1748)*, Band I, Basel: Birkhuser, 531 pp., esp. 235–236, edited by Otto Spiess. This work the originals of the L'Hospital-Bernoulli correspondence and a detailed account of the relations between them. It also has a complete list of L'Hospital's publications. The quotation above comes from pp. 97–98; the translation from Morris Kline (1972–??), *Mathematical Thought from Ancient to Modern Times* (1972), Oxford, p. 473.

SECTION IX.

Solution de quelques Problémes qui dépendent des Méthodes précédentes.

PROPOSITION I.

Problême.

163. SOIT *une ligne courbe* AMD (AP = x, PM = y, AB = a) *telle que la valeur de l'appliquée y soit exprimée par une fraction, dont le numérateur & le dénominateur deviennent chacun zero lorsque* x = a, *c'est à dire lorsque le point* P *tombe sur le point donné* B. *On demande quelle doit être alors la valeur de l'appliquée* BD. FIG. 130.

Soient entendues deux lignes courbes *ANB*, *COB*, qui ayent pour axe commun la ligne *AB*, & qui soient telles que l'appliquée *PN* exprime le numérateur, & l'appliquée *PO* le dénominateur de la fraction générale qui convient à toutes les *PM*: de sorte que $PM = \frac{AB \times PN}{PO}$. Il est clair que ces deux courbes se rencontreront au point *B*; puisque par la supposition *PN* & *PO* deviennent chacune zero lorsque le point *P* tombe en *B*. Cela posé, si l'on imagine une appliquée *bd* infiniment proche de *BD*, & qui rencontre les lignes courbes *ANB*, *COB* aux points *f*, *g*; l'on aura $bd = \frac{AB \times bf}{bg}$, laquelle * ne différe pas de *BD*. * *Art. 2.* Il n'est donc question que de trouver le rapport de *bg* à *bf*. Or il est visible que la coupée *AP* devenant *AB*, les appliquées *PN*, *PO* deviennent nulles, & que *AP* devenant *Ab*, elles deviennent *bf*, *bg*. D'où il suit que ces appliquées, elles mêmes *bf*, *bg*, sont la différence des appliquées en *B* & *b* par rapport aux courbes *ANB*, *COB*; & partant que si l'on prend la différence du numérateur, & qu'on la divise par la différence du dénominateur, après

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the many papers of Leibniz and the Bernoulli's in *Acta eruditorum* and the *Journal des sçavans* generated a great deal of enthusiasm for the new calculus, but if you were not a genius of their stature, these papers were impenetrable. Consequently l'Hospital's book was a great success. Seldom has a book been so well received for it truly filled a need, something few textbook writers can say of their books today. It is not surprising then that the book went through numerous reprintings and editions in the eighteenth-century. L'Hospital's name appeared on the title page of the second edition of 1715, which appeared after his death. It was even translated into English in 1730 by E. Stone (indicating that the rift between the English and continental mathematicians was not that strong), but Newton's fluxional notation was used in this edition.[4]

L'Hospital did make other interesting contributions to mathematics, especially in his posthumous book on the conics, *Traté analytique des sectiones coniques* (Paris, 1707), but none were so great as the expository work in his *Analyse*.[5]

In the preface to his book, just after mentioning Leibniz and especially "the young professor at Groningen," i.e., Bernoulli, L'Hospital admitted "I have made free use of their discoveries, so that I frankly return to them whatever they please to claim as their own." He does not spell out precisely what he learned from them, but of course it was common knowledge that Leibniz had published the first papers on the calculus and that Bernoulli had tutored L'Hospital. He does explicitly state that the foundations of the calculus given in the book are his own discovery, and he was also instrumental in getting Leibniz to make statements about his own foundational views.

After Johann Bernoulli received a copy in Groningen, he wrote to L'Hospital praising the book as admirably done, praising the arrangement of the propositions, and praising the intelligibility of the exposition, even thanking him for mentioning his name, and promising to return the favor in his next publication. These ingratiating comments seem out of character for Bernoulli, but there is truth in what he says: L'Hospital's exposition was deserving of praise. It is a wonderful book.

Immediately after L'Hospital's death in February 1704, Bernoulli published a generalization of L'Hospital's rule which allowed for its repeated application. In this paper, in the August issue of *Acta eruditorum*, he complained that L'Hospital had not given him ample credit and then laid public claim to the most novel and interesting result in the book, the theorem in §163 that we now call L'Hospital's rule. [Who instituted this name? When?] As Bernoulli was not noted for either modesty or generosity, and as he had already been involved in more than his share of priority disputes, this claim was generally dismissed by his contemporaries. While Bernoulli is a difficult character to defend, it should be said that his claim was only made after L'Hospital's friend Saurin implied that the rule was due to Leibniz.[6]

[4] For information on the various editions see Bernoulli, op. cit., 1955.

[5] Julian Lowell Coolidge (1873–1954), "Guillaume L'Hospital, Marquis de Sainte-Mesme," pp. 147–170 in his *The Mathematics of Great Amateurs* (1949), Oxford: Clarendon Press. Dover reprinted this in 1963. This chapter is the best source of information about the entire corpus of L'Hospital's work. However, sometimes Coolidge is so brief as to be incomprehensible.

[6] "Perfectio Regula sua edita in Libro Gall. *Anal-*

The question of priority was only answered in this century when Johann Bernoulli's *Lectiones de calculo differentialium* [Lectures on the Differential Calculus] were published in 1922 by Paul Schafheitlin (1861–1924).[7] They were written in 1691–1692 when Bernoulli was in France tutoring L'Hospital and they contain considerable overlap with the material that appeared in L'Hospital's *Analyse des infiniment petits* of 1696. This publication raised the plagiarism issue again after two centuries, but did not clear things up entirely, for the historian Carl Boyer, writing a quarter of a millenium after L'Hospital's book appeared, still agreed with Eneström's 1894a opinion that "the broad claims of Bernoulli with respect to the authorship of the material are not substantiated."[8]

The situation was not cleared up completely until 1955 when Bernoulli's correspondence, including that with L'Hospital, was published. It contains a most unusual letter that L'Hospital, then in Paris, wrote to Bernoulli in his home town of Basel on 17 March 1694:

> I shall give you with pleasure a pension of three hundred livres, which will begin on the first of January of the present year, and I shall send two hundred livres for the first half of the year because of the journals that you have sent, and it will be one hundred and fifty livres for the other half of the year, and so in the future. I promise to increase this pension soon, since I know it to be very moderate, and I shall do this as soon as my affairs are a little less confused. . . . I am not so unreasonable as to ask for this all your time, but I shall ask you to give me occasionally some hours of your time to work on what I shall ask you and also to communicate to me your discoveries, with the request not to mention them to others. I also ask you to send neither to M[onsieur]. Varignon nor to others copies of the notes that you let me have, for it would not please me if they were made public. Send me your answer to all this and believe me, Monsieur tout à vous
> le M. de Lhospital

Bernoulli's response to this letter has been lost, but we know from his letter of 22 July 1694 that he accepted L'Hospital's proposal of a regular salary in exchange for help on mathematical problems and, implicitly, for providing L'Hospital with results to publish under his own name. We do not know how long this arrangement lasted, but Bernoulli's finances improved, and, as mentioned earlier, he soon had a position in Groningen. Considering Bernoulli's effusive praise of L'Hospital's book, it is likely that the agreement was still in effect in 1696. At any rate, the agreement prevented Bernoulli from laying claim to L'Hospital's rule while he was still alive. As we have seen though, he staked his claim as soon as L'Hospital died.

We can learn several interesting things from this letter. Both men kept a copy of the lectures which Bernoulli wrote while he was in France in 1691–1692. These lectures were the first exposition of the new calculus of Leibniz, and they are an excellent presentation. There were also lectures on the integral calculus which L'Hospital planned publish, but did not when he learned that Leibniz has similar intentions. Unfortunately, these plans of Leibniz, like many of his plans, came to nought. Bernoulli's lectures on the integral calculus were published within his own lifetime in his *Opera omnia* of 1742.[9] They contained a footnote that he had also written on the differential calculus

yse des infiniment petits, Art. 163. pro determinando valore fractionis, cujus Numerator & Denominator certo casu evanescunt," *Acta eruditorum*, August 1704, pp. 375ff; reprinted as No. LXXI in his *Opera omnia*, vol. 1, pp. 401–405. The example that Bernoulli gives here provides a useful classroom example:

$$y = (a\surd(4a^3 + 4x^3) - ax - aa)$$
$$: (\surd(2aa + 2xx) - x - a).$$

Bernoulli asks for the value when $x = a$; we seek the limit on y as x tends to a.

[7] *Die Differentialrechnung von Johann Bernoulli aus dem Jahre 1691–1692* (1924), Ostwald's Klassiker #211, 56 pp (Engelmann, Leipzig). Translation of 1922a edited by P. Schafheitlin. Reviewed by H. Wieleitner, *Isis*, vol. 5 (1923), pp. 186–7.

[8] Carl Benjamin Boyer (1906–1976), "The first calculus textbooks," *Mathematics Teacher*, 39 (1946), 159–167, especially p. 163. This is very useful, but remember it was written before Bernoulli 1955 appeared. It discusses the contents of L'Hospital's book.

[9] Johann Bernoulli, "Lectiones mathematicae, de methodo integralium, aliisque. Conscriptae in usum ill. Marchionis Hospitalii, cum auctor parisiis ageret, annis 1691 & 1692," pp. 385–558 of vol. 3 of his *Opera omnia*, four volumes, Lausanne and Geneva 1742; reprinted Hildesheim: Georg Olms, 1968. These lectures were written in Paris in 1691–

in 1691–1692, but that similar work was published by L'Hospital in his 1696 book. Unfortunately, it was too late for these lectures to have a significant effect on the development of the integral calculus.

The most important thing that we learn from the correspondence is that Bernoulli's claim to being the discoverer of L'Hospital's rule is correct, for in his letter of 22 July 1694 the rule appears along with two examples. One of these examples appears in L'Hospital's *Analyse des infiniment petits*, the other appears there in slightly modified form.

Now that we know that L'Hospital's rule is due to Johann Bernoulli, the question arises as to whether we should rename it. Of course not! For one thing, we could never change such a well established tradition in mathematics. Perhaps the only benefit of doing so would be that we would not have students talking about "The Hospital Rule"; instead they would refer to "The Bernoulli Rulie." Seriously though, this is an excellent example of the principle that one cannot lay claim to a scientific discovery until one has given it to the scientific community. Since it was L'Hospital who gave the rule to the world, he should get the credit. Besides, he bought it fair and square.

The Original Statement of L'Hospital's Rule

Without doubt the most famous section of L'Hospital's *Analyse* is §163, which contains the first printed statement and proof of the famous rule:

163 Proposition I. *Let AMD [Fig. 130] be a curve ($AP = x$, $PM = y$, $AB = a$) of such a nature, that the value of the ordinate y is expressed by a fraction, the numerator and denominator of which, do each of them become 0 when $x = a$, viz. when the point P coincides with the given point B. It is required to find what will be the value of the ordinate BD.*[10]

L'Hospital next gives a proof of the result and then follows it with two examples. We shall analyze the proof shortly, but first let us see his examples and make a few comments about them. The first asks for the evaluation of

$$y = \frac{\sqrt{2x^3 - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{x^3}}$$

when $x = a$. The solution provided is quite similar to what our students do today and so will not be reproduced [do it as an exercise]. He computes the quotient of the differentials and then sets $x = a$. Note that derivatives are not used; that concept is still in the future—see Grabiner 1983a. Naturally no limits are used here for that concept had to wait for Cauchy in the nineteenth century.

A Conjecture: It should be clear that this is a rather weird first example. Certainly I would not ordinarily begin with such a complicated example in class, although since this is "the first example," I do mention it as part of the curious history of L'Hospital's rule. Years ago I thought that this example must have arisen from some physical problem; this was when I still was inclined to believe the Kline-May thesis that all good (important, interesting, ... —add enough modifiers to make this statement true) mathematics arises out of real world problems. But I was unable to find any such origin for this example. Thus I concluded that Bernoulli simply concocted this example for use in the "classroom" with L'Hospital. Recently I found this same example in a letter from Bernoulli to L'Hospital [1955, p. 232]. In this letter Bernoulli mentions this expression in the same sentence in which he mentions the Florentine Enigma of Vincenzo Viviani (1622–1703): cut four congruent windows from a hemisphere in such a way that the area remaining is quadrable, i.e., the integral for this area can be evaluated in elementary (hopefully algebraic) terms. Unfortunately, the passage is quite obscure and I am unable to tell whether he is just listing two separate problems that he is working on or means to imply that they are closely connected. I rather think that he means the latter, for the two examples are

92 for the use of L'Hospital. See the footnote on p. 387 where he states that his lectures on the differential calculus were published by L'Hospital.

[10] Dirk J. Struik (born 1894), "L'Hôpital. The analysis of the infinitely small," pp. 312–316 of his *A Source Book in Mathemtics, 1200-1800*, Harvard University Press, 1969. Struik's "The origin of L'Hôpital's rule," *Mathematics Teacher*, 56 (1963), 257–260 is very informative. It quotes the letter of 17 March 1694 from L'Hospital to Johann Bernoulli from Bernouli 1955. Stuik's "The origin of L'Hospital's rule," pp. 435–439 in NCTM's Thirty-First Yearbook, *Historical Topics for the Mathematics Classroom*, 1969 contains information here that is not in either of the above. For information about Struik, see David E. Rowe, "Interview with Dirk Jan Struik," *The Mathematical Intelligencer*, 11 (1989), no. 1, pp. 14–26

separated by a semicolon. The second example of L'Hospital is considerably simpler:

Example 2. Evaluate the following expression when $x = a$.

$$y = \frac{aa - ax}{a - \sqrt{ax}}$$

Curiously, L'Hospital does not write out this example using his new rule, but comments that "We could solve this example without need of the calculus of differentials in this way." He solves for the radical, squares to eliminate it, and then divides by $x - a$. Next he sets $x = a$. This is the old math. It would fit nicely into his last chapter where he compares the new methods with the old. Bernoulli's second example would do even better. This old technique, as well as its revision by Leibniz and Newton was later soundly — and rightly — criticized by Berkeley.

L'Hospital simplified this example a bit from that given by Bernoulli in his letter of 22 July 1694 (1955, p. 236):

$$y = \frac{a\sqrt{ax} - xx}{a - \sqrt{ax}}$$

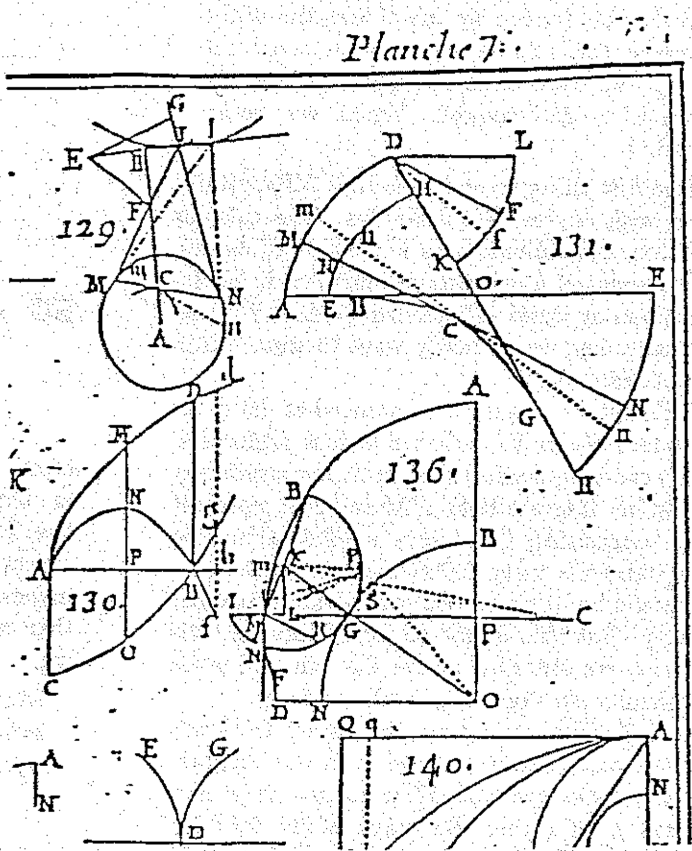
Now Bernoulli gives this example "for verifying the method" that he has just sent to L'Hospital. He points out that it can be worked out using the "common geometry" of Descartes that was well known at the time. Pedagogically this is a beter way of using the example; we should show our students what can (and cannot) be done using older methods and thus use these examples to justify newly introduced techniques. This example is harder to verify using the old method, for you obtain a quadratic equation when eliminating the roots [It is not clear to me why only one root is considered]. Thus we see that L'Hospital simplified the example, but not used it so cleverly.

These two examples are the only ones given, and L'Hospital's rule is applied to just one of them. It is as if L'Hospital were saying, "OK, that's it; if you don't get it with these two examples, too bad; it's time to move on to the next section." While I am not advocating greater speed in our courses today, perhaps there is a pedagogical point here that we should heed. There is no need to give numerous examples in order to get the idea across. We concentrate too much on computational technique and not enough on the underlying ideas.

Perhaps you are surprised that there are no examples dealing with exponentials, logarithms, and the trigonometric functions. The explanation is historical: Johann Bernoulli was doing the first work on calculus of exponentials at that very time. The calculus of the trigonometric functions, and indeed the trigonometric functions themselves (*qua* functions), were not developed until the 1730s when Euler did so.

The Proof of L'Hospital's Rule

Figure 130 is on a plate at the back of the volume, but we reproduce part of it here for the student's convenience and amusement:



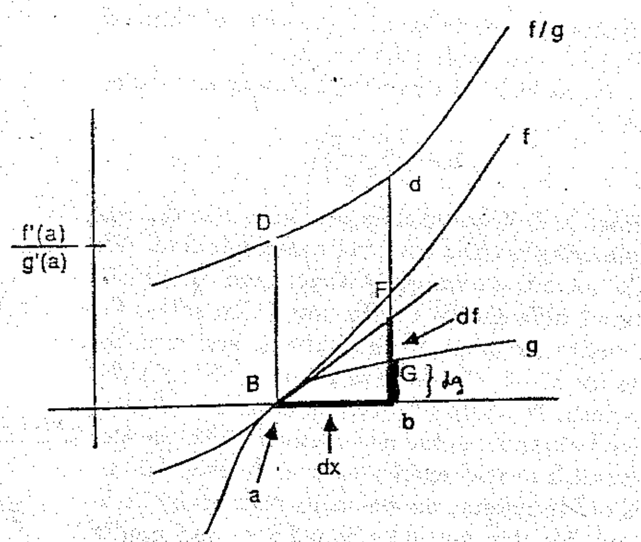
Next we shall give the proof in L'Hospital's book:

> Let ANB, COB, be two curves (having the line AB as a common axis) of such a nature, that the ordinate PN expresses the numerator, and the ordinate PO the denominator of the general fraction representing any ordinate PM: so that $PM = (AB x PN)/PO$. Then it is manifest, that these two curves will meet one another in the point B; since by the supposition PN, PO do each become 0 when the point P falls in B. This being supposed, if an ordinate bd be imagined infinitely near to BD, cutting the curves ANB, COB in

the points f, g; then will $bd = (ABxbf)/bg$, which will be equal to BD. Now our business is only to find the relation of bg to bf. In order thereto it is manifest, when the abscissa AP becomes AB, the ordinates PN, PO will be 0, and when AP becomes Ab, they do become bf, bg. Whence it follows, that he said ordinates bf, bg, themselves, are the differentials of the ordinates in B and b, with regard to the curves ANB, COB; and consequently, if the differential of the numerator be found, and that be divided by the differential of the denominator, and having made $x = a = Ab$ or AB, we shall have the value of the ordinates bd or BD sought. Which was to be found.[11]

The first thing to realize is that L'Hospital is dealing with curves, not functions. The calculus of Newton and Leibniz was a calculus of curves; Euler introduced a calculus of functions in his *Introductio in analysin infinitorum* (1748). For presentation today, we certainly want to state this in terms of functions.

L'Hospital's diagram is somewhat hard for the modern reader to decipher in that L'Hospital (and his contemporaries) does not know which way is up. In his diagram both PM and PO represent positive quantities. The curve ANB is above the axis AB and the curve COB is below, so the quotient should be below in our way of doing analytic geometry. But this is not so for L'Hospital. His quotient curve AMD is above the axis. To avoid this difficulty we shall redraw the picture:



$$BD \approx bd = \frac{bF}{bG} \approx \frac{df}{dg} = \frac{df/dx}{dg/dx} = \frac{f'(a)}{g'(a)}$$

In L'Hospital's diagram the curve ANB was above the axis, but we moved it in our diagram so that it looks more familiar to us. It is now the curve f, or rather the graph of the function f. The curve COB has been relabelled g. Now suppose the two curves f and g which meet at the point a, where both are 0. The quotient f/g is not defined for $x = a$, but it is this value that L'Hospital wishes to determine. What L'Hospital is saying is that

$$BD \approx bd = \frac{bf}{bG} \approx \frac{df}{dg} = \frac{df/dx}{dg/dx} = \frac{f'(a)}{g'(a)}.$$

Thus he provides a very nice proof of the rule. It is much more intuitive than the proofs that are customarily given today using Cauchy's extended mean value theorem.[12] Admittedly, infinitesimals are used in the proof, but, after the work of Abraham Robinson in the 1960s we need no longer be concerned about the rigor of infinitesimal techniques. In my view, what is lost in rigor is more than made up for in intuitiveness and understanding.

Moral: The original proofs are often much more understandable than modern ones.

[11] Struik, *Source Book*, p. 316.

[12] Yes, I am perfectly aware that this proof admits of counterexamples. It does not work when $bG = 0$. More precisely, if the function in the denominator is 0 arbitrarily close to a, then the proof fails. But, in the seventeenth-century, there were no such functions. Today, if one of my students comes to me with a function such as $\sin(1/x)$, then we are ready to do some more mathematics.

This file, L'Hospital.tex, was typeset using TeX on May 29, 1996 at 2:21 P.M.

The Integral Sign

One of the greatest treasures for the historian of mathematics is the notes that Leibniz made as he was discovering the calculus. We can follow the details of his invention in an English translation in *The Early Mathematical Manuscripts of Leibniz* (1920), edited by J. M. Child. In the manuscript dated October 29, 1675 Leibniz wrote:

> Utile erit scribi $\int$ pro omn. ut $\int l$ pro omn. l, id est summa ipsorum l. [It will be useful to write $\int$ for omn., as $\int l$ for omn. l, that is, the sum of these l's.][1]

A little later in the same manuscript he indicated that he was aware that he was creating a new field of mathematics when he wrote:

> Satis haec nova et nobilia, cum novum genus calculi inducant. [These are sufficiently new and notable, since they will lead to a new calculus.]

The integral symbol itself was the long form of the letter S which was frequently used by Leibniz at the time in his ordinary writing. Previously he had used the notation of Cavalieri: ***omn.l***, for *omnes linea*, that is, for the sum of the lines.[2] Cavalieri's notation had also been used by Mengoli and Fabri.

In the same manuscript Leibniz wrote $\int x^2 = x^3/3$.[3] Leibniz also used the characteristic triangle in this manuscript, though both Pascal and Barrow had used it before him.

The integal sign first appeared in print eleven years later in 1686 in Leibniz's first paper on the integral calculus, "De geometria recondita et analysi indivisibilium atque infinitorum" [On a deeply hidden geometry and the analysis of indivisibles and infinities], which appeared in *Acta eruditorum*[4], a journal which Leibniz was instrumental in founding. In this paper the bottom half of the symbol was amputated so that it looked more like our letter f. It is interesting that one of the things that Leibniz did in this paper was to give an equation of the cycloid involving the integral sign.

The paper dealt with John Craig's little book *Methodus figurarum lineis rectis et curvis comprehensarum quadraturas determinandi*, [Method of Determining the Quadratures of figures Bounded by Curves and Straight Lines], London 1685. A Scottish theologian who loved to apply mathematics to matters divine, Craig (1660?–1731) was around Cambridge about 1680 and so knew of Newton's work on the calculus. The book refers to Leibniz's work on the calculus and uses his differential notation (being the first work in England to do so), so this explains why Leibniz was reviewing it. The stimulus for Leibniz writing this paper was that Craig had attributed to Leibniz a paper that was actually written by Tschirnhaus. Leibniz wanted to correct any false impressions concerning his work and so he sent this paper on the integral calculus to *Acta eruditorum*. In it he showed that quadratures were a special case of the inverse tangent problem. It is curious that Leibniz's work on the calculus was referred to in a book published in England before Newton published on the calculus.[5]

Leibniz originally spoke of the integral calculus as the *calculus summatorius*, a name obviously connected with summation [the sign Σ was introduced by Euler in 1755]. He first saw the word "integral" in a paper of Jakob Bernoulli in the May 1690 *Acta eruditorum* = *Opera*, 421–426, wherein Bernoulli solved the isochrone problem which Leibniz had posed to the public in 1687:

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1 Cajori, *A History of Mathematical Notations*, vol. 2, §570, p. 203.

2 J. E. Hofmann, *Leibniz in Paris*, p. 188.

3 A photo of this is in Cajori (*Notations*, vol 2, p. 243; This facsimile is reproduced from C. I. Gerhardt's *Briefwechsel von G. W. Leibniz mit Mathematikern* (1899). See §§544, 570, and especially §620 of Cajori.

4 5 (1686), 292–299. The integral sign appeared first on p. 297. The original is reprinted in his *Mathematische Schriften*, Abth. 2, Band III, 226–235. A German translation is in Ostwald's *Klassiker*, No. 162, and the relevant passage, in English translation, is in Struik's *Source Book* (where the original date of the manuscript is misprinted). See Aiton 1985a, pp. 119 and 168.

5 See Aiton, *Leibniz. A Biography* (1985), p. 119.

Find the curve such that when a point falls along this curve the vertical component of its velocity is constant. Later the younger Bernoulli brother, Johann, reduced the problem to the simple differential equation $dy \cdot \sqrt{y} = dx \cdot \sqrt{a}$, and then "integrated" to obtain $\frac{2}{3}y\sqrt{y} = x\sqrt{a}$, which is a semicubical parabola. To see the details of how Johann Bernoulli presented this problem in lectures to his pupil L'Hospital in 1691-1692 see *Die erste Integralrechnung*, Ostwald's *Klassiker*, no. 194 (1914), pp. 142–143.

Leibniz tried, in the year 1695, to persuade Johann Bernoulli, Jakob's younger brother, to use his terminology of "sums":

> I leave it to your deliberation if it would not be better in the future, for the sake of uniformity and harmony, not only between ourselves but in the whole field of study, to adopt the terminology of summation instead of your integrals. Then for instance $\int y\,dx$ would signify the sum of all y multiplied by the corresponding dx, or the sum of all such rectangles. I ask this primarily because in that way the geometrical summations, or quadratures, correspond best with the arithmetical sums or sums of sequences. ... I do confess that I found this whole method by considering the reciprocity of sums and differences, and that my considerations proceeded from sequences of numbers to sequences of lines or ordinates.[6]

This request of Leibniz provided Bernoulli with the opportunity to explain the origin of the term integral:

> Further, as regards the terminology of the sum of differentials I shall gladly use in the future your terminology of summations instead of our integrals. I would have done so already much earlier if the term integral were not so much appreciated by certain geometers [a reference to French mathematicians, especially l'Hôpital, who had studied Bernoulli's *Integral Calculus*] who acknowledge me as the inventor of the term. It would therefore be thought that I rather obscured matters if I indicated the same thing now with one term and now with another. I confess that indeed the terminology does not aptly agree with the thing itself (the term suggested itself to me as I considered the differential as the infinitesimal part of a whole or *integral*; I did not think further about it).[7]

Leibniz reluctantly accepted Bernoulli's suggestion to call his new invention the integral calculus, but everyone else quickly accepted it.

Limits of integration were at first indicated only in words. Euler was the first to use symbols for limits of integration. He did this in his three volume work on the integral calculus, *Institutiones calculi integralis* (1768–1770). Fourier in his 1822 *La Théorie analytique de la chaleur* used $\int_a^b$, although he had used it previously in print in 1819. In 1823 F. Sarrus first used the signs $\left[F(x)\right[_a^x$ and $\left[_a^x F(x)\right.$ to indicate the process of substituting the limits a and x in the general integral $F(x)$. Cauchy and Moingo also later used this notation, but several others were suggested before the mathematical community settled on the variant of Sarrus's notation that we use today.[8]

Perhaps it will be amusing to mention that there are several facetious[9] origins of the integral sign. Franklin M. Turrell, in a paper entitled "The definite integral symbol,"[10] notes that "if an apple is peeled by hand with a knife, beginning at the stem end and circling about the central axis without breaking the peel until the opposite end is reached, a regular spiral is obtained which forms an elongated S when placed on a flat surface with the inside of the peel up." With illustrations and scholarly references to the apple literature, Turrell conjectures that the apple peel suggested the integral symbol to Leibniz. In a later paper[11] with the same title, Martin G. Beumer relates that around the turn of the century when the calculus "began to penetrate the circle of the adepts of physical chemistry" the 1901 Nobel prize winner in Chemistry, Jacobus Henricus van't Hoff (1852–1911) gave the following derivation of the word integral: "The word is derived from *Integer* (whole, entire) and *Aal* (German word for: eel)."

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6 Leibniz to Johann Bernoulli, February 28, 1695; quoted from Bos, *AHES*, 14 (1974), 21.

7 Johann Bernoulli to Leibniz April 30, 1695; Bos, ibid, 21–22.

8 See Cajori's *A History of Mathematical Notations*, §625–629 for details.

9 "Abstemious" and "caesious" are the only other English words I know that contains all five vowels in alphabetical order.

10 *American Mathematical Monthly*, 67 (1960), 656–658.

11 *Monthly*, 81 (1974), 1095–1096.

Fermat's Integration of Powers

The 1657 publication of the *Arithmetica infinitorum* (Arithmetic of Infinities) of John Wallis (1616–1703) prompted Pierre de Fermat (1601–1665) to compose a treatise on the quadrature methods he had been developing for two decades. The result was his posthumous *De aequationum localium transmutatione et emendatione* ... (On the transformation and alteration of local equations for the purpose of variously comparing curvilinear figures among themselves or to rectilinear figures, to which is attached the use of geometric proportions in squaring an infinite number of parabolas and hyperbolas). This was probably written in 1658, but not published (or circulated) until 1679 in his *Varia opera mathematica*, so it appeared too late to have a profound effect on the development of the calculus. Our interest is in the section dealing with the quadrature (or finding the area under) the "higher hyperbolas," $x^p y^q = k$, and "higher parabolas," $y^p = kx^q$, which Fermat created.

Fermat's ingenious integration of $y = x^n$ provides a very interesting example which can easily and profitably be done in class. When introducing the integral via the notion of Riemann sums, the problems quickly become too hard. Here is a neat trick that will allow you to use the definition to evaluate the integrals of powers of x in class—and for an arbitrary integer n. The only fact that is needed is the sum of the geometric series, a fact the student needs in other situations anyway.

The clever idea that Fermat had was not to divide the interval $[0, a]$ of integration into equal subdivisions, but rather to use unequal subdivisions. It is clear where this idea came from. He had been finding the area under his generalized hyperbolas $y = 1/x^n$ on the interval $[1, \infty]$, and in this situation it is natural to use equal subdivisions. But when he considered the generalized parabolas $y = x^n$ on $[0, 1]$, it was natural to invert the ray $[1, \infty]$ into the finite interval $[0, 1]$. When this is done, unequal partitions are the reasonable choice to make. So now let's see what Fermat did.

Let E be a positive constant less than 1. Use it to divide the closed interval $[0, a]$ into infinitely many subintervals of different lengths at the points $\ldots, aE^3, aE^2, aE, a$. Then construct rectangles on these subintervals so that they circumscribe the curve $y = x^n$ and add up their areas, which form a geometric progression:

$$\begin{aligned} S_E &= \sum_{i=0}^{\infty} (aE^i)^n (aE^i - aE^{i+1}) \\ &= a^{n+1} \sum_{i=0}^{\infty} E^{in} E^i (1 - E) \\ &= a^{n+1}(1-E) \sum_{i=0}^{\infty} (E^{n+1})^i \\ &= a^{n+1}(1-E) \frac{1}{1 - E^{n+1}} \\ &= \frac{a^{n+1}}{1 + E + E^2 + E^3 + \ldots + E^n}. \end{aligned}$$

The last step here follows by elementary algebra. Now as E approaches 1 we see that S_E approaches $a^{n+1}/n + 1$. Thus we have

$$\int_0^a x^n \, dx = \frac{a^{n+1}}{n+1}.$$

This argument works for positive integers. Do you see where it fails when $n = 1/2$? Fermat was able to extend his idea to include all rational values of n except the logarithmic case, $n = -1$.

The above proof is quite easy for us to understand, but this is primarily because we have translated it into modern notation and nomenclature. Fermat used proportions in his argument so it is fairly difficult to understand. A translation error in [4] (line 21 on page 220 should be "increasing," not "decreasing") makes the original argument even harder.

EXERCISES

1. There are very few problems where unequal subdivisions are useful, but here is one. Use the definition of the derivative to show

$$\int_0^a \sqrt{x} \, dx = \frac{2}{3} a^{2/3}.$$

This file, fermat.tex, was typeset using TeX on May 29, 1996 at 2:54 P.M.

Use the n partition points $x_k = bk^2/n^2$ and the right endpoints of these intervals as evaluation points. [From George F. Simmons, *Calculus with Analytic Geometry*, McGraw Hill, 1985, p. 176.]

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[2] Mahoney, Michael S., *The Mathematical Career of Pierre de Fermat (1601–1665)*, Princeton University Press, 1973. See especially pp. 243–253.

[3] Mahoney, Michael S., "Fermat, Pierre de," *Dictionary of Scientific Biography*, 4, 566–576.

[4] Struik, Dirk J., "Fermat. Integration," in *A Source Book in Mathematics, 1200–1800*, Harvard University Press, 1969, pp. 219–222.

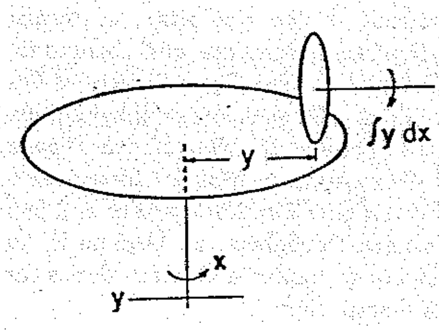
The Bush Differential Analyzer

In the 1920s electrical engineers were mathematically sophisticated despite having learned their calculus from texts that stressed graphical presentation and intuition. Given this background, it is not surprising that they were frustrated by the second order differential equations which they met in their research, for many of them had no closed form solutions. Vannevar Bush of the Electrical Engineering Department at MIT built an elegant "solution," the differential analyzer. Portions of this machine are part of collection of the Smithsonian's National Museum of American History. Photographs can be found in Owens 1986a and Goldstine 1972a.

The most sophisticated version of Bush's differential analyzer was first demonstrated on December 13, 1941. "Weighing almost a hundred tons and comprising some two thousand vacuum tubes, several thousand relays, a hundred and fifty motors, and automated input units, the analyzer was the most important computer in existence in the United States at the end of the war." [Owens 1986a, 63]. Throughout World War II the analyzer labored over the computation of ballistic tables and the profiles of radar antennas. While it is true that this analog computer was the most sophisticated computer of any kind in operation during the war, it soon lost out in favor of digital computers. In fact this was so obvious to the research team working on the analyzer that in 1946 they took the totally unheard of step of returning $50,000 in grant money to the Rockefeller foundation and informing them that to continue research on analog computers would be not only unjustified but "foolish" [Owens 1986a, 85]. For an interesting discussion of the complex social and technological issues that doomed the analog computer after World War II see Owens 1986a.

Bush's differential analyzer was a dynamic mechanical computer which could be reconfigured to model — although sometimes with great difficulty — a variety of differential equations. Although it could not be built until the invention of the torque amplifier (in 1927 by the engineer C. W. Nieman) provided a way to eliminate slippage, its main component was a modification of the integraph invented by James Thomson (1822–1892) in the 1860s but published only in 1876 after his brother William, better known as Lord Kelvin, discussed his idea of a tide-calculating machine. (This is the same James Thomson who coined the word "radian.")

The basic integraph consists of two disks on the ends of perpendicular shafts. The first disk is mounted horizontally. The second is vertical with its circumference resting on the horizontal disk at a variable distance $y = f(x)$ from its center. If the rotation of the vertical shaft has a constant speed of x, then that of the horizontal shaft is proportional to the integral of f. This device provides a nice simple model of the definite integral.



Of course this same device can be used to differentiate functions. If the speed of rotation of the horizontal shaft is proportional to $g(x)$ and that of the vertical shaft is proportional to x, then the vertical wheel will be forced to stay at a distance dg/dx from the center of the horizontal wheel. Thus we have a nice analog illustration of the fundamental theorem of the calculus.

The analyzer had "a very considerable educational value," as Bush wrote in 1928 and I am sure it still does if we teachers of calculus would only discuss it in class. When we do, "one part at least of formal mathematics will become a live thing" and the student will learn to "think straight in the midst of complexity." [Owens 1986a, 85–86.]

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The first planimeter was designed by the German engineer J. M. Hermann about 1814 but no description of it was published and so it was soon forgotten. The first publication on planimeters was by Gonell in 1825 and was based on Hermann's model [Horsburgh 1914a, p. 190]. The most famous person to be associated with planimeters was the physicist James Clerk Maxwell (1831–1879) who invented one in 1855. James Thomson's was based on Maxwell's [Goldstine 1972a, p. 40]. It is not entirely clear whether Bush rediscovered the wheel and disk integrator or learned of it from the papers of William and James Thomson [Owens 1986a, 68, 93; Goldstine 1972a, 92]. For a detailed discussion of the various integrators and planimeters see Horsburgh 1914a.

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1986a "Vannevar Bush and the Differential Analyzer: The Text and Context of an Early Computer," *Technology and Culture*, 27(1986) 63–95.

This file, bushdiffanalyzer.tex, was typeset using TEX on May 29, 1996 at 3:09 P.M.

Saint Vincent and the Logarithm

Gregorius Saint Vincent (1584–1667) was born in Belgium, studied theology, philosophy, and mathematics in Rome with Christoph Clavius (1537–1612), and was ordained a Jesuit priest in 1613. After Galileo announced his telescopic discoveries in his *Siderius nuncius* (1610), Saint Vincent hinted that he supported the heliocentric system.[1] This made him suspect to his Catholic superiors, and so he was forced to teach Greek and mathematics in various cities around Europe. His most famous work, *Opus geometricum quadraturae circuli et sectionum coni* (Geometrical works on the quadature of the circle and the conic sections) was written in the 1620s, but the Jesuits refused to let him publish it then. When he fled Prague in 1631 just ahead of the advancing Swedes, he was forced to leave the manuscript behind. He did not see it again until 1641. Finally, with the help of several students, it was published in 1647.

The volume is huge—containing more than 1250 large pages. It contains the first presentation of the summation of infinite geometric series, a method of trisecting angles using infinite series, and the result Saint Vincent considered his most important: a method for squaring the circle. Unfortunately, this result was incorrect, as Huygens first pointed out in 1651. Although this error destroyed his reputation, the work contains much of value which influenced Leibniz, Wallace, and Wren.

The frontispiece of the *Opus geometricum* is, without a doubt, the most magnificent allegory in all of mathematical publishing. In the foreground, Archimedes, who was killed by a Roman soldier in 212 B.C. during the sack of Syracuse, is drawing the diagram for his proof of the area of a circle[2]. Cowering attentively behind him is Euclid, who is looking on in awe. The character anachronistically wearing swim goggles[3] has not been identified. Wading in the estuary is Neptune, whose banner carries the slogan "Plus ultra," there is *more beyond* this ancient geometry, yet the ancients are prevented from getting there by the Pillars of Hercules.[4] But Gregorius has discovered this new land of mathematics—at least,

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1 This is the most exciting scientific book ever written. I encourage you to read the *Sidereus Nuncius or the Sidereal Messenger by Galileo Galilei*, translated and edited by Albert Van Helden, University of Chicago Press, 1989.

2 *Measurement of a Circle*, Proposition I: "The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle." See Heath, *The Works of Archimedes* (1897), p. 91; reprinted by Dover.

3 Eyeglasses were invented in 1285 by Allesandro de Spina.

4 "In the word histories section [of the *Random House English Language Desk Reference*], there is a nice background to the verb *non plus*, which means "to make utterly perplexed"; it goes, "The original Latin phrase was 'non plus ultra,' meaning 'no more beyond' and allegedly inscribed on the *Pillars of Hercules*, beyond which no ship could safely sail." ...the "pillars" are the rocky promontories flanking the Strait of Gibralter, and as we

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this frontispiece claims so. In the background the sunbeam carries the words "Mutat quadrata rotundis" (the square is changed into a circle) which are illustrated by the putto holding the square frame which focuses the sunbeam into a circle on the ground. Note that
the putti are tracing it out with a compass, and that the circle is correctly drawn in perspective as an ellipse[5].

The copy of the title page and frontispiece of the *Opus geometricum* which are reproduced here come from the copy in the United States Military Academy Library at West Point, which contains a very extensive collection of older mathematical works (a bibliography of which is under preparation).[6]

The most important aspect of this book for the calculus is a surprising connection between the natural logarithm and the rectangular hyperbola, $xy = 1$. Let $A_{a,b}$ denote the area above the interval $[a, b]$ and below the hyperbola $y = 1/x$. We prefer not to use integrals to describe this area so as not to prejudge the issue; indeed we are talking about some proto-calculus that was done before the time of Newton and Leibniz. Now let x_i, $i = 1, 2, \ldots, n$, partition the interval $[a, b]$ into equal pieces. Then Gregorius bounded the area by

know, there is nothing beyond there except the Atlantic Ocean and the place where you fall off the earth." So wrote William Saffire in "On Language: Gifts of Gab," *New York Times Magazine*, December 3, 1995, p. 38. Alas, the dictionary is wrong about people believing that one could fall off the earth. The flat earth myth was created by Washington Irving (1783–1859) in his romantic biography *History of the Life and Voyages of Christopher Columbus* (1828). See my "How Columbus encountered America," Mathematics Magazine, 65 (1992), 219–225.

[5] The interpretation of this engraving is primarily my own. The only description of this frontispiece that I am aware of is "A curious mathematical title-page," by Florian Cajori, *The Scientific Monthly*, 14 (1922), 293–295.

[6] Note that the work is signed by René François de Sluse (1622–1685), who developed a method for finding tangents to algebraic curves just before Newton (1642–1727) discovered his own. The volume also contains notes which, I conjecture, were written by Sluse.

ANTVERPIÆ. APVD IOANNEM ET IACOBVM MEVRSIOS. ANNO M.DC.XLVII.

lower and upper "Riemann" (1826–1866) sums:

$$\sum_{i=1}^{n} \frac{b-a}{nx_i} \leq A_{a,b} \leq \sum_{i=1}^{n} \frac{b-a}{nx_{i-1}}$$

If we do the same for the interval $[ta, tb]$, partitioning it at the points tx_i, $i = 1, 2, \ldots, n$, then we obtain, in a similar fashion, the sums:

$$\sum_{i=1}^{n} \frac{tb-ta}{ntx_i} \leq A_{ta,tb} \leq \sum_{i=1}^{n} \frac{tb-ta}{ntx_{i-1}}$$

It is apparent that the ts cancel in these sums. Since both $A_{a,b}$ and $A_{ta,tb}$ are bounded by the same sums, they must be equal. Today we would use a limiting process here, but that was not Saint

P. GREGORII

A S^{to} VINCENTIO

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ET SECTIONVM CONI

Decem libris comprehensum.

Vincent's way; he used a proof by exhaustion—to use a phrase that he coined in this book—although the method goes back to Archimedes.

A student of Gregorius, Alfonzo Antonio de Sarasa noted[7] that area is additive and if we put $a = 1$ then we have

$$\begin{aligned} A_{1,xy} &= A_{1,x} + A_{x,xy} \qquad (1) \\ &= A_{1,x} + A_{1,y} \qquad (2) \end{aligned}$$

where (2) follows by the property proved by Gregorius. Then he made a most interesting remark. He said that this area behaved like a logarithm, for the rule

$$A_{1,xy} = A_{1,x} + A_{1,y}$$

is completely analogous to

$$\ln(xy) = \ln(x) + \ln(y)$$

To us this appears to be only a tiny step, but what looks like a small step to us may not have been small to the creative mathematician who made it. It is very easy to read things into a text, and as historians we must avoid it. What is obvious to us may not be obvious to the person who wrote it.[8]

Our way of introducing the logarithm, i.e., by *defining* it to be

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

was not proposed for use in the schools until the end of the nineteenth-century, when Felix Klein (1849–1925) did it in his *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis*, Dover, p. 156; The German original is from 1908.

This proof of Gregorius is a very interesting way of developing the logarithm. I have used it in class and found it to be quite satisfactory.[9]

[7] *Solutio problematis a M. Mersenno propositi* (Solution of a Problem Proposed by Mersenne), 1649. This work is not at West Point, but it is bound at the end of the copy of the *Opus geometricum* in the New York Public Library.

[8] We always say that the logarithm is a transcendental function, and this was certainly known to Leibniz and the Bernoulli's, but I do not know who proved it. There is a nice proof by R. W. Hamming, of error correcting code fame, in a paper entitled "An elementary discussion of the transcendetal nature of the elementary transcendental functions," *American Mathematical Monthly*, 77 (1970), pp. 294–297; reprinted in *A Century of Calculus, Part II 1969–1991*, edited by Tom M. Apostol et al., MAA 1992, pp. 80–83.

[9] Rosemary Schmalz, "A 'natural' approach to e," *The Mathematical Gazette*, vol. 74, #470, December 1990, pp. 370-372 outlines a similar approach without mentioning Gregorius.

Torricelli's Trumpet

An Infinite Solid With Finite Volume

I do not remember this of Torricellio ...
to understand this for sense, it is not required
that a man should be a geometrician or a logician,
but that he should be mad.

Thomas Hobbes[1]

Evangelista Torricelli (1608–1647) showed such promise as a youth that he was sent to study with the Camaldolese monk, Benedetto Castelli (1578–1643), a mathematician, hydraulic engineer, and student of Galileo (1564–1642). A treatise Torricelli wrote on projectile motion attracted Galileo's attention, so he was able to live with Galileo in Florence during the last few months of Galileo's life. Bonaventura Cavalieri (1598–1647) and Vincenzo Viviani (1622–1703) were also part of the circle of Galileo at this time, so they were all friends. After Galileo died[2], Torricelli was appointed to Galileo's position of mathematician and philosopher to the Duke of Tuscany.

In 1641 Torricelli showed that an infinite solid could have finite volume. He thought he was the first to discover this, but he may have been anticipated by his contemporaries Fermat and Roberval, and certainly by was Oresme in the fourteenth century. The result was published in 1644 in *Opera geometrica*, the only work Torricelli published in his lifetime. Publication caused a sensation. In addition, Marin Mersenne (1588–1648), who had met Torricelli while visiting Italy in 1645, quickly spread the word. The best mathematicians of the seventeenth century were amazed and perplexed by this result, just as our students should be. Mathematical intuition is not always inherent; it takes time to develop.

Consider the rectangular hyperbola $y = 1/x$, which has been truncated at some point, say $(1/2, 2)$ to be specific. Then add a horizontal line segment from $(0, 2)$ to $(1/2, 2)$ to form a new curve. This curve is to be rotated around the x-axis and we wish to find the volume of the resulting solid.

Consider an arbitrary point (x_0, y_0) on the hyperbola, and draw a horizontal line from it to $(0, y_0)$. When this line segment is rotated around the x-axis a cylinder of radius y_0 and height x_0 is generated. This cylinder has lateral surface area $(2\pi y_0)x_0$. Since the point (x_0, y_0) lies on the hyperbola, the surface area of the cylinder is 2π, which is constant, and so independent of the choice of the point on the hyperbola.

Torricelli now forms a circle of area 2π, the surface area of each of the cylinders, and places it in the horizontal plane $y = y_0$. This is repeated for each point on the hyperbola. The result is a stack of circles which form a cylinder with base area 2π and height 2. The volume of this cylinder is 4π. By Cavalieri's Principle, this is the volume of the Trumpet of Torricelli[3].

In this proof, Torricelli has extended Cavalieri's Principle to allow curved indivisibles (not infinitesimals). To see that Cavalieri's original statement applies, think of the solid as made up of concentric horizontal cylinders. Then cut these cylinders up through the x-y-plane to the x-axis. Then unroll the cylinders so that they form rectangles that live in the same horizontal plane as the top of each cylinder. It is in this horizontal plane that Torricelli places a circle of equal area.

This result of Torricelli is incredible. It is the most interesting and ingenious application of Cavalieri's Principle. He has found a solid, that is infinitely long, and yet has finite volume. Moreover, his construction is remarkably simple. There seems to be no end to what clever people can do. Yet, this technique is *ad hoc*; it is crucial that the area of each horizontal cylinder be the same, for otherwise he could not have constructed the vertical cylinder.

In 1658 Christiaan Huygens (1629–1695) and René-François de Sluse (1622–1685) found an even

[1] *The English Works of Thomas Hobbes*, vol. 7, p. 445.

[2] Galileo is buried in a magnificent tomb to the left of the narthex in the church of Santa Croce in Florence. Across the nave, the tomb of Michelangelo (1475–1564) is of similar grandure. This indicates that scientists and artists held equal social positions in Florence. Would that it were the same today.

[3] A neologism of Jan van Maanen.

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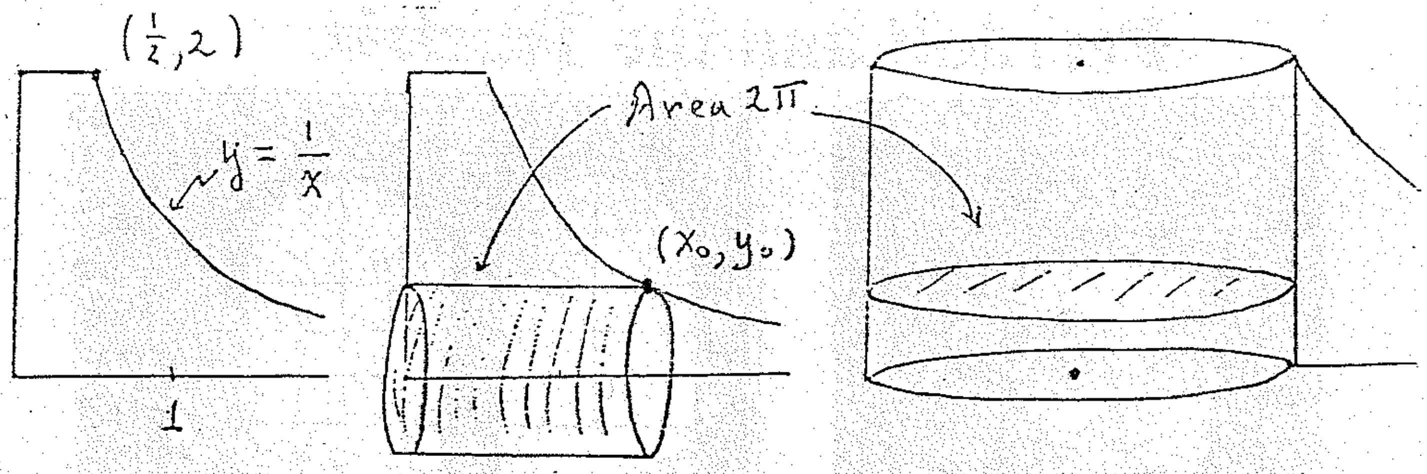

more remarkable solid, one with finite surface area and infinite volume. Sluse, in a letter to Huygens, proudly described it as a drinking glass, that had a small weight, but that even the hardiest drinker could not empty ("levi opera deducitur mensura vasculi, pondere non magni, quod interim helluo nullus ebibat."[4]).

Exercises:

1. Evaluate the integral $\int_{1/2}^{\infty} \pi(\frac{1}{x})^2\,dx$ and compare the value with the volume of Torricelli's Trumpet. Explain why the answers disagree.
2. Find the surface area of Torricelli's Trumpet.
3. The first time I presented this in class, I commented that the result was *ad hoc* and probably would not generalize. My students observed that if $y = 1/x^n$, $n > 1$, is rotated about the x-axis, then the volume is finite because it fits into the cylinder. This is a physical realization of the comparison test for integrals that I had never seen before. Are there any cases where you can obtain an exact result, say by making the solid a cone? [This is a serious question; I don't know the answer.]
4. Show that when the portion of $y = x^{-3/4}$ that lies above the ray $[1, \infty)$ is rotated aroung the x-axis, the resulting solid has finite volume and infinite surface area. The integral for surface area is not elementary, so you will need to use the comparison test to show that it is infinite.
5. View the film "Infinite Acres."
6. If the region that lies above the inteval $(0, 1]$ and below the graph of $y = \sqrt{x}$ is revolved about the x-axis, a solid called "Thor's anvil" is generated. Show that it has invinite volume and that the region generating it has finite area. Examples of this type seem not to be in our calculus books. [Dawson]

Sources:

This note is based on: (1) The *Dictionary of Scientific Biography*, (2) Carl Boyer, *The History of the Calculus and Its Conceptual Development*, pp. 125–126, (4) John W. Dawson, Jr., "Contrasting examples in improper integration," *The Mathematics Teacher*, March 1990, 201–202, and (4) Jan A. van Maanen, "Alluvial deposits, conic sections, and improper glasses, or history of mathematics applied in the classroom," pp. 73–91 in *Learn From the Masters*, edited by Frank Swetz, et alia, MAA, 1995. Boyer cites "De solido hyperbolico acuto" (Concerning pointed hyperbolic solids), which appears in *Opera di Evangelista Torricelli* (1919), vol. I (of 3), part I, pp. 173–221. Van Maanen cites Christiaan Huygens, *Œuvres Complètes*, vol. 2, pp. 164, 168, 212 and vol. 14, pp. 199, 200, 306–312, and also *Correspondance du P. Marin Mersenne*, vols., 12–14. these original sources. Margaret E. Baron, *The Origins of the Infinitesimal Calculus* (1969), which has been reprinted by Dover, gives a good treatment of Torricelli's work on pp. 182–194.

After this note was written a detailed discussion of the philosophical importance of this result was published by Paolo Mancosu and Ezio Vailanti as "Torricelli's infinitely long solid and its philosophical reception in the seventeenth century," *Isis*, 82 (March 1991), 50–70.

[4] Huygens, *Œuvres Complètes*, vol. 2, p. 168.

Perrault and the Tractrix

The problem is to find the path of an object pulled by an inelastic string by someone walking along a straight line (of course, the pulled object should not be on the straight line). It was posed in 1676 to Leibniz, who was in Paris inventing the calculus, by the physician, anatomist, and architect Claude Perrault (1613–1688). If the name Perrault sounds familiar, it is probably because you have seen it on the title pages of such Mother Goose fairy tales as *Cinderella*, *Puss' n Boots* and *Little Red Riding Hood*.[1] These delightful stories were written by Claude's younger brother Charles Perrault (1628–1703). Claude Perrault was trained as a physician and quietly practiced for twenty years before he was invited, probably through the intervention of his brother, to become a founding member of the Académie des Sciences in 1666. There he took an interest in many of the scientific problems of the day, earning a reputation for the careful and detailed anatomical descriptions that he published. In 1667 he joined a committee responsible for the design of the entrance façade of the Louvre. Perrault also designed the Paris Observatoire which Colbert hoped would be the center of the Academy's activities (see the figure; Perrault is next to the king; develop this more; cite article in Physics Journal). He died of an infection incurred while dissecting a camel.

Claude Perrault is remembered for his annotated translations of Vitruvius's *De architectura* (1673 and 1684), and for a work on the design of columns which was influential throughout the eighteenth century. Perrault's only contribution to mathematics, as far as I am aware, was to pose the problem that concerns us here. It was all the rage in mathematical circles of the day Perrault used to illustrate it by placing his watch in the middle of a table and pulling the end of its watchchain along the edge of the table.[2]

At this point I would like to put in a plug

[1] For a mathematical translation see "The story of LR^{2}H," *Mathematics Magazine*, ???

[2] Biographical information drawn from the *Dic-*

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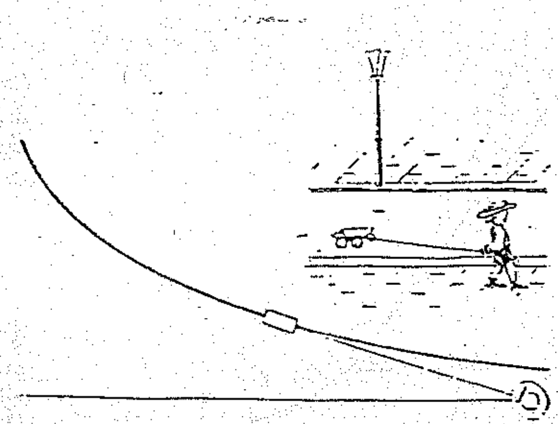
for biography, for it fleshes out history. For some years I tended to downplay the importance of biography, but I have come to appreciate the role that it plays in helping us to understand mathematics. For example, the simple fact that Descartes traveled to The Netherlands where he met van Schooten, does a lot to simplify the complicated history of analytic geometry (e.g., it explains why it was Schooten who published a Latin edition of Descartes' *Geometrie*). What interests students most is anecdotes, but that is only a small part of what they learn from our presentation of the larger picture. I don't mean to downplay anecdote, for a catchy line will be remembered by the student, and if they remember the mathematician, they have a better chance of remembering the mathematics.

The image of Perrault and his camel is too vivid to forget. Another line that I have used with great effect, when talking of falling bodies, is from Jacob Bronowski: "Galileo was a short, square, active man with red hair, and rather more children than a bachelor should have"[3] For a most interesting paper on this topic, I suggest you carefully consider Helena M. Pycior's "Biography in the mathematics classroom" in *History in Mathematics Education* (1987), edited by Ivor Grattan-Guinness. But I have digressed.

Perrault's question was one of the earliest inverse tangent problems. These are problems where some characterization of the tangent to a curve is given with the goal to find the curve itself; today we call them differential equations. The first inverse tangent problem was posed by Debaune in 1638. While Debaune's problem is often part of a differential equations course, it is a little too complicated for the usual calculus course. However, Perrault's problem is an ideal example in calculus.

I like to begin my classroom presentation with a delightful picture from the first English edition of *Mathematical Snapshots* by Hugo Steinhaus. It shows a young boy pulling a little wagon on a string. Note that the top view even shows the ribbon on his hat. He is walking along the sidewalk and the wagon is in the street. This picturesque way of viewing the problem gives the curve its name. From the Latin "trahere," which means "to drag or pull," comes our word tractrix (When we extract roots of equations we go back to the same Latin root). This is not the only name of



the curve. "The Dutch physicist Huygens called the tractrix the "dog curve" because it resembles the curve described by the nose of a dog being dragged reluctantly on a leash."[4]

The most important mathematical constraint in this problem is that the length of the string is a constant. In the seventeenth century this was expressed by saying that the (length of the) tangent is constant; we have an "equitangential curve." This way of speaking sounds strange to us, since we think of the tangent line as the whole line, and hence not having finite length, but in those days, before they had a good grasp of such elementary parts of analytic geometry as the equations of lines, they viewed the tangent as the line segment on our tangent line between the point of tangency on the curve and the x-axis.

Leibniz solved the problem at once but unfortunately no trace of his original work survives.[5] Newton also solved the problem, but again no worksheets survive.[6] Priority in publication goes to Huygens, in a letter to Henri Basnage de Beauval

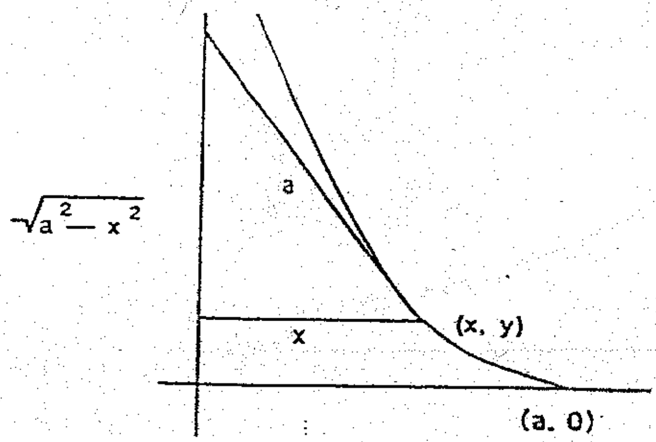
tionary of Scientific Biography* and the *Encyclopedia of World Art*.

[3] *The Ascent of Man*, p. 200.

[4] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries. Development and History*, second edition, 1980, p. 327.

[5] *Acta eruditorum*, September 1693.

[6] *The Mathematical Papers of Isaac Newton*, 3, p. 26, note 31.



(1656–1710) of 1693(2?).[7] I do not know when the problem first became a textbook problem. In fact, I know of no research whatsoever on the issue of when certain problems entered our calculus textbooks. This would be a most interesting line of research for someone to take up.

One aspect of this problem that I particularly like is that the student is forced to grapple with the geometry of the situation in setting up the differential equation to solve. Students need to be given many opportunities to model real world problems, for this skill is much more important than the formal calculus techniques which dominate our books.

Suppose the wagon is originally at the point $(a, 0)$, and the young boy starts at the origin, and then walks up along the y-axis. (The student should be asked why I rotated the picture.) Now sketch the curve:

[7] *Histoire des Ouvrages des Sçavans* for December 1692/February 1693 = Huygens, *Œuvres complètes*, 10, pp. 407–409 and 418–420.

Draw the tangent at an arbitrary point (x, y) on the tractrix, complete the triangle as in the diagram, and remember that the tangent has length a. From the picture, the slope of the tangent line is

$$\frac{\sqrt{a^2 - x^2}}{-x}$$

(reflecting the picture in the x-axis would eliminate both the minus sign, and a little lesson for the student), and by definition the slope is dy/dx, so we have the equation

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

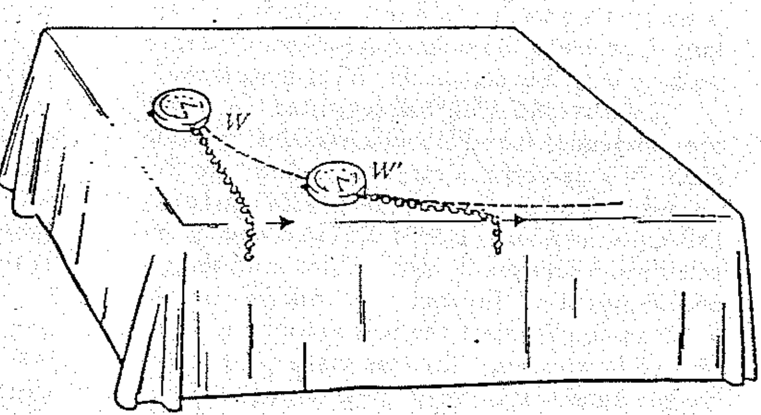
The hard part is now finished (this always surprises students when I say it, but it's true). Now all we have to do is integrate. The substitution $t^2 = a^2 - x^2$ reduces this to an algebraic integral which can be evaluated using partial fractions. (The trigonometric substitution $u = a\sin(x)$ is slightly harder to work out.) Any student who has had some experience with these basic integration techniques has no trouble finding the equation of the tractrix:

$$y = \sqrt{a^2 - x^2} - a\ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right).$$

I have used this as a classroom example and also as a problem on a take home exam. When it was an exam question a little of the history was included in the statement of the problem, and I presented the rest later in class. In both cases Perrault's tractrix problem was well received.

Related exercises for the students, which are well within their grasp, ask for the curve of constant subnormal, or the curve of constant subtangent. Of course, you will have to explain that the subnormal of a point on a curve is the line segment from the abscissa of the point to the place where the normal intersects the x-axis.

An interesting property of the tractrix is that it is the evolute of a catenary. The subject of involutes and evolutes, which arose in the work of Huygens on pendulum clocks and played a large role in L'Hospital's calculus text of 1696, is no longer mentioned in our books, but would be a good way to introduce more geometry. Again Steinhaus has a nice way of describing this. To form the catenary hang a chain on a wall. Then cut the chain at the bottom and let the chain fall under the action of gravity. The path traced out by either end of the chain is a tractrix. To keep the chain from falling out of the catenary shape as it unwinds, put some nails on the concave side of the chain. This can also be demonstrated in class by cutting a catenary out of a piece of plywood and then having a string with a piece of chalk at the end which initially is at the vertex of the catenary. Keep the string taut as you unwind it from the catenary and you will obtain a nice sketch of the tractrix. It is better to take a physical model into class and spend the time drawing an accurate curve than it is to just sketch it out freehand while explaining in words why it works. Students will remember the model, but not your sketch. Fortunately, I am blessed with a colleague, Cliff Long, who enjoys making such models and is willing to make them for me.



If the tractrix is rotated about its asymptote, an infinite horn-shaped surface of constant negative curvature is formed. A patch of the surface can be moved to any other place without stretching it; you only have to bend it as you do when you move a patch of a cone. Every point on the surface is a saddle point. Ernest Ferdinand Adolf Minding (1806–1885) discovered this property of the surface and so called it an "anti- sphere."[8] The surface was used by Eugenio Beltrami (1835–1900) because it isisometric to part of the hyperbolic plane. He also introduced the name pseudosphere which we now use for the surface. Huygens showed that the volume of the pseudosphere is one-half the volume of a sphere of radius a, where a is the length of the tangent line that generates the tractrix. To check this is a challenge.

As a more real world application, we note that Schiele used the pseudosphere as the shape of a bearing with considerable longitudinal thrust.[9] Here again we have seen another advantage of using a historical approach. We can mention theoretical developments and real world applications that would probably not otherwise be well received by the students, and would seem out of place without the history.[10]

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[8] H. Steinhaus, *Kalejdoskop Matematyczny*, 1956, p. 309; this remark is not in the English editions of Steinhaus.

[9] For details see E. H. Lockwood, *A Book of Curves* (1961), p. 124.

[10] An earlier version of this note appears in my paper "My favorite ways of using history in teaching calculus," pp. 123–134 in *Learn From the Masters*, edited by Frank Swetz et alia, MAA, 1995.

This file, Perrault-Tractrix.tex, was typeset using TeX on May 29, 1996 at 8:54 P.M.

After I did this problem in class, I said that it was just a warm up for a more difficult and very famous problem, the catenary problem. This time we have only the main cable. What is its shape?

Galileo (1564–1642) had suggested that a heavy rope suspended from both ends would hang in the shape of a parabola, a conjecture which was disproved by Joachim Jungius (1587–1657) and published posthumously in 1669; Huygens had an unpublished refutation in 1646. If you read the geometric proof of Huygens,[2] you will see what a great accomplishment the new calculus of Leibniz and Newton was. The true shape of the curve was not known until 1690/91 when Huygens, Leibniz, and Johann Bernoulli (1667–1748) replied to a challenge of Jakob Bernoulli (1654–1705). The name "catenary" was introduced by Huygens in a letter to Leibniz in 1690; it derives from the Latin "catena," which means "chain." This was the first independent work of Johann Bernoulli, who was immensely proud that he had solved the catenary problem (Acta eruditorum, 1691; 2, vol. 1, 48–51) and that his brother Jakob, who had posed it, had not. Writing to Pierre Remond de Montmort (1678–1719) years later, on 29 September 1718, Johann boasted:

> The efforts of my brother were without success; for my part, I was more fortunate, for I found the skill (I say it without boasting, why should I conceal the truth?) to solve it in full and to reduce it to the rectification of the parabola. It is true that it cost me study that robbed me of rest for an entire night. It was much for those days and for the slight age and practice I then had, but the next morning, filled with joy, I ran to my brother, who was still struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more to try to prove the identity of the catenary with the parabola, since it is entirely false. The parabola indeed serves in the construction of the catenary, but the two curves are so different that one is algebraic, the other is transcendental.[3]

After giving all of this history to my class, time had run out, so I announced that we would omit the derivation of the equation of the catenary. To my great surprise, the students howled in protest and insisted that we do the derivation next time. Naturally, I was happy to oblige, but this event was so unique that I have ever since attributed it to the fact that I had presented the problem in its historical setting. I have no stronger example of history as a motivating force.

So now let us derive the equation of the catenary. The notation in the figure, with a few changes, will suffice. Of course, the roadway is no longer there. This time the weight is that of the cable alone. It is distributed uniformly along the cable, not uniformly in the horizontal direction. This is the main difference from the suspension bridge. The derivation becomes more complicated since we must introduce the parameter s, which denotes arc length. Consequently, the weight of the cable is ρs. The horizontal forces are the same as before, so we have

$$T(0) = T(x)\cos(\alpha),$$

but the downward vertical force is ρs, so this time we have

$$\rho s = T(x)\sin(\alpha).$$

Eliminating $T(x)$ between these two equations and then solving for s we have $s = k\tan(\alpha)$, where $k = T(0)/\rho$. If we differentiate x and y with respect to arc length s we have:

$$dx/ds = \cos(\alpha) \quad \text{and} \quad dy/ds = \sin(\alpha).$$

By the chain rule (what could be more fitting to use in this problem),

$$\frac{dx}{d\alpha} = \frac{dx}{ds}\cdot\frac{ds}{d\alpha} = \cos(\alpha)\cdot k\sec^2(\alpha) = k\sec(\alpha)$$
$$\frac{dy}{d\alpha} = \frac{dy}{ds}\cdot\frac{ds}{d\alpha} = \sin(\alpha)\cdot k\sec^2(\alpha) = k\sec(\alpha)\cdot\tan(\alpha).$$

Integrating each of these we have

$$x = k\ln|\sec(\alpha)+\tan(\alpha)| \quad \text{and} \quad y = k\sec(\alpha).$$

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[2] H. J. M. Bos, "Huygens, Christiaan," *Dictionary of Scientific Biography*, volume 6, pp. 597–613, especially p. 601.

[3] *Der Briefwechsel von Johann Bernoulli, (1667–1748)*, edited by Otto Spiess, 1955, pp. 97–98; translation from Morris Kline, *Mathematical Thought from Ancient to Modern Times*, 1972, p. 473.

The integral

Finally, if we expend a little effort and eliminate α from these two equations we obtain $y = k\cosh(x/k)$, which is the equation of a catenary.[4]

Thus we have seen that when a suspension bridge is being erected, and only the cable is up, then it assumes the shape of a catenary. However, when the roadway is installed below, then the cable changes shape to a parabola.

Perhaps this is an opportune point to mention the issue of historical accuracy in the classroom. Contrary to the professional historian of mathematics, the classroom teacher need not be a slave to historical details and methods. The teacher should not lie, but it is not necessary to tell the whole story. To provide an overabundance of detail will bore the students and will not advance our goal of using history to motivate and instruct the students. In particular, it is not necessary, and seldom desirable, to use the same methods to derive results that their inventors did. The above derivation for the catenary is stated in modern language, and I would certainly not apologize for doing so in class.[5]

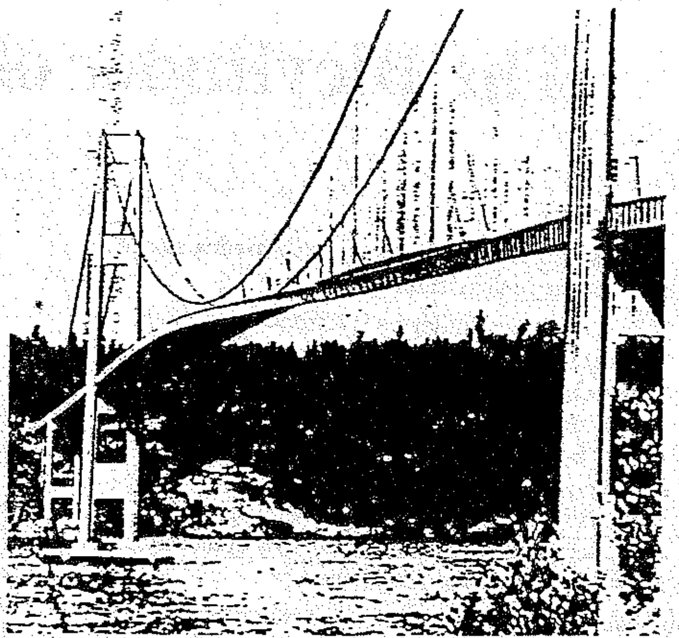

Figure 1. Burt Farquharson's classic photograph of the fatal twisting of the Tacoma Narrows Bridge under the action of wind. (Photograph courtesy of the Special Collections Division, University of Washington Libraries.)

Figure 2. Kármán vortex street, a pattern of alternating vortices created by the confrontation of a nonstreamlined body and an airstream and initially suggested as the source of the periodic impulses that drove the Tacoma Narrows Bridge to collapse.

[4] For a somewhat different solution see C. H. Edwards Jr. and David E. Penney, *Calculus and Analytic Geometry*, 1982, pp. 371–373.

[5] An earlier version of this note appears in my paper "My favorite ways of using history in teaching calculus," pp. 123–134 in *Learn From the Masters*, edited by Frank Swetz et alia, MAA, 1995.

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The Bicylinder of Archimedes

In the preface to his *Method*, Archimedes stated the following proposition:

> If in a cube a cylinder be inscribed which has its bases in opposite squares and the surface of which touches the four other faces, and if in the same cube another cylinder be inscribed which has its bases in other squares and the surface of which touches the four other faces, the solid bounded by the surfaces of the cylinders, which is enclosed by the two cylinders, is two-thirds of the whole cube.[1]

Unfortunately, the proof of this proposition is not included in the only extant manuscript of the work, so all we can do is to conjecture how Archimedes proved the result.[2]

Instead of looking at the work of Archimedes, let us turn to the East. The Chinese mathematician Liu Hui is best known for the *Haidao suan-jing* (The Sea Island Mathematical Manual)[3] of AD 263. This volume contains nine problems involving surveying that we would solve today using trigonometry. Liu, to use only his surname, tried unsuccessfully to find the volume of the bicylinder, which he called a "box-lid." After he failed, he wrote the following poem[4] about his efforts:

> Look into the cube
> And outside the box-lid;
> Though the dimension increases,
> It doesn't quite fit.
> The marriage preparations are complete;
> But square and circle wrangle,
> Thick and thin are treacherous plots,
> They are incompatible.
> I wish to give my humble reflections,
> But fear that I will miss correct principle;
> I dare to let the doubtful points stand,
> Waiting
> For one who can expound them.

Duane W. DeTemple[5], who coined the word "bicylinder," showed an elementary computation of its volume using Cavalieri's Principle. The idea is to compare the top half of the bicylinder with the top half of the circumscribing cube, from which a square based pyramid has been removed (with its base at the top of the circumscribing cube and apex at the center). If a horizontal plane intersects the semi-bicylinder at height z above the horizontal x-y-plane, then that intersection is a square the length of one side of which is $2\sqrt{r^2 - z^2}$. Hence that square has area $4(r^2 - z^2)$. Now the corresponding cut from the circumscribed cube with square pyramid removed is a 'square-annulus' of area $(2r)^2 - (2z)^2$. Thus the two solids have the same volume.

It is worth remarking that the volume of a hemisphere can be computed by Cavalieri's principle in an entirely analogous way: Compare it with the circumscribed cylinder from which a cone has been removed.

Problems

1. Find the volume of the wedge which is cut from the base of a right circular cylinder by a plane passing through a diameter of the base and inclined at an angle of 45° to the base.

History: This problem first appeared in the *Method* of Archimedes, who states it in a somewhat more general form: A right circular cylinder

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[1] E. J. Dijksterhuis, *Archimedes*, p. 314.

[2] For Zeuthen's reconstruction of the proof see "The Method of Archimedes," which is a supplement (with new pagination) to T. L. Heath, *The Works of Archimedes*, pp. 48-51.

[3] See Ang Tian Se and Frank Swetz, "A Chinese mathematical classic of the third century: The *Sea Island Mathematical Manual* of Liu Hui," *Historia Mathematica*, 13(1986), 99–117.

[4] Quoted from *Calculus* by Deborah Hughes-Hallet et al., preliminary edition, 1992, p. 531.

[5] "An Archimedean property of the bicylinder," *The College Mathematics Journal*, 25(1994), 312–314. DeTemple also gives an elementary and very clever derivation of the surface area of the bicylinder. Because the sides of the bicylinder are lined surfaces, it is possible to make a model of the figure from four sinusoidal lenses of cardstock. DeTemple begins his article by recollecting the story of the tomb of Archimedes. He ends: "Can anyone refer me to a good tombstone engraver? No hurry, of course."

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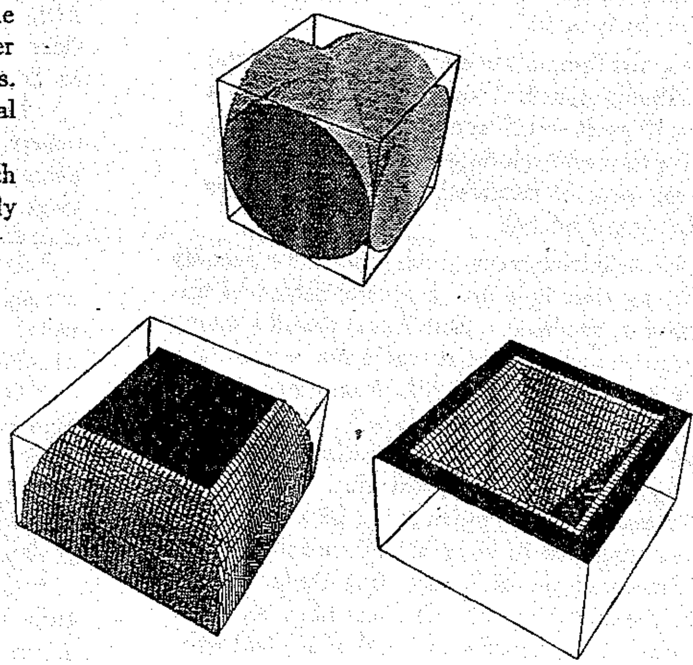
is inscribed in a right prism with a square base and a plane is passed through one edge of the top of the prism and then through a diameter of the base; the top edge uniquely determines which diameter of the base the plane passes through. Then Archimedes gives an argument using his method of indivisibles that shows the volume of the wedge is one-sixth of the volume of the prism.

2. The axes of two right circular cylinders, each of radius a, intersect at right angles. Find the common volume.

History: This problem also first appeared in the *Method* of Archimedes. In his introductory letter to Eratosthenes, Archimedes mentions that he had earlier sent several results, "urging you to find the proofs," and that now he intends to send the proofs. The first mentioned result is this one, the second that in the problem above.

Archimedes states the result in a different way than we do today. He starts with a cube and inscribes two right circular cylinders with orthogonal axes. The result is that the part cut off by the cylinders is two-thirds of the whole cube. The reason for this way of stating the result is that the Greeks did not actually find volumes, but rather found the ratios of the volumes of various solids. This use of proportion avoided the use of irrational quantities.

Unfortunately, the proof of this result, which was to be Proposition 15, is missing from the only surviving manuscript.



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Schwarz's Paradox

After receiving his doctorate from the University of Turin in 1880, Giuseppe Peano (1858–1932) became the assistant (a new position at Turin designed to give the best students an entry into the academic world) to one of the mathematics professors. During the the 1881-1882 academic year, Peano was the assistant of Angelo Genocchi (1817–1889) who held the chair of infinitesimal calculus and who had been Peano's calculus teacher. Genocchi, who was 65 years old, had to ask Peano to teach his class because of his failing health. Calculus was a second year course, with four classes and two recitations per week. When the topic of surface area came up, Peano naturally turned to the commonly accepted definition of Joseph Alfred Serret (1819–1885) of the area of a surface S bounded by a curve C:

> area is the limit of the elementary areas of the inscribed polyhedral surfaces P bounded by a curve Γ as $P \longrightarrow S$ and $\Gamma \longrightarrow C$, where this limit exists and is independent of the particular sequence of inscribed polyhedral surfaces which is considered.[1]

This definition seems quite plausible especially considering that it is just a generalization of the definition of arc length. But Peano found a counterexample to show that it would not work. In a class lecture of May 22, 1882, Peano gave a corrected definition of surface area.[2] This was the first of a long sequence of discoveries that Peano made regarding elementary calculus.

Naturally, he was quick to tell his old teacher, but was surprised to learn that Genocchi was already aware of the difficulty. But Genocchi had not discovered it himself. On December 20, 1880, Hermann Amadeus Schwarz (1843-1921) had written Genocchi informing him of the difficulty in Serret's definition.[3] After Peano told Genocchi of his discovery, Genocchi wrote his friend Charles Hermite (1822–1901) of the independent discoveries of Schwarz and Peano because Hermite had used the incorrect definition of Serret in his lectures. Hermite wrote Schwarz for details so that he could revise his course. Schwarz wrote up the details of his discovery and Hermite quickly included them in the second edition of his mimeographed lecture notes (1883).[4] This did not please Schwarz for he did not feel right in publishing a result that had already appeared in print. The originnl note of Schwarz was not published until the second volume of his collected works appeared in 1890.[5] In the meantime, Peano published his work in 1890.[6] Although Peano's work was published first, it was clear to all concerned that Schwarz had priority for this near simultaneous discovery.[7]

In an amazing example of simultaneous discovery, both Schwarz and Peano found the same example to show that Serre's definition will not work. They considered an ordinary cylinder of

[1] J. A. Serret, *Cours de calcul différential et intégral*, 2 vols., first edition 1868, p. 296, second edition 1880, p. 293. Translated, with commentary by Axel Harnack, *Lehrbuch der Differential- und Integralrechnung*, 2 volumes, 1884-85. Translation from Lamberto Cesari, *Surface Area*, Princeton University Press, 1956, pp. 24–25. Pp. 24–26 of Cesari contain an excellent treatment of the Schwarz paradox.

[2] See p. 143 of his lithographed course notes.

[3] U. Cassina, "L'area di una superficie curva nel carteggio ineditio di Genocchi con Schwarz ed Hermite," *Rend. Ist. Lomb. Sci. Lett.*, (3) 83 (1950), 311-328.

[4] C. Hermite, *Cours professé à la Faculté de Sciences* Paris, first edition 1881-82, second edition 1883.

[5] H. A. Schwarz, "Sur une definition erronée de l'aire d'une surface courbe," *Gesammelte mathematische Abhandlungen*, II, pp. 309–311, 369–370.

[6] G. Peano, "Sulla definizione dell'area d'una superficie," *Atti della Accademia Nazionale dei Lincei, Rendiconti, Classe di scienze fisiche, mathematiche e naturali*, (4) 6 (1890), 54-57. English translation as "On the definition of the area of a surface (1890, 1903)," pp. 137–142 of *Selected Works of Giuseppe Peano*, translated and edited by Hubert C. Kennedy, University of Toronto Press, 1973. The '1903' refers to Peano's very interesting work, *Formulaire mathématique*, volume 4, where his example is given.

[7] Hubert C. Kennedy, *Peano. Life and Works of Giuseppe Peano*, Dordrecht: D. Reidel, 1980, pp. 7-10.

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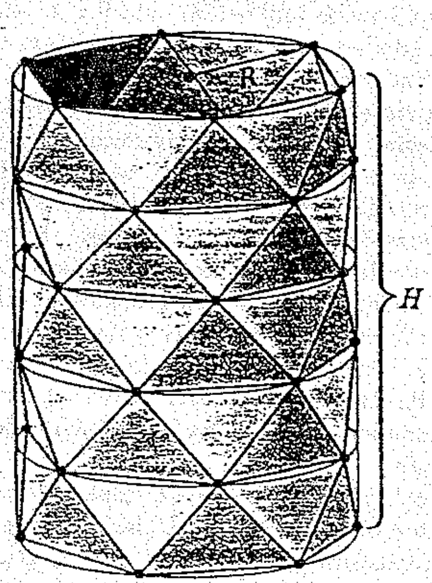
height H and radius R (Peano took $H = R = 1$.). Clearly the lateral surface area of this cylinder should be $2\pi RH$. What Schwarz and Peano did was to construct a sequence of inscribed polyhedral surfaces whose areas tend to any number not less than $2\pi RH$.

Divide the lateral area of the cylinder into n horizontal strips each of height H/n by using a sequence of parallel horizontal circles. Take m equally spaced points around each of these circles, but doing so in such a way that the points on each circle are midway between those on the circle above. By joining every pair of adjacent points on one circle to the point midway between on the circle below (and above), a polyhedral surface consisting of mn congruent isosceles triangles will be formed.

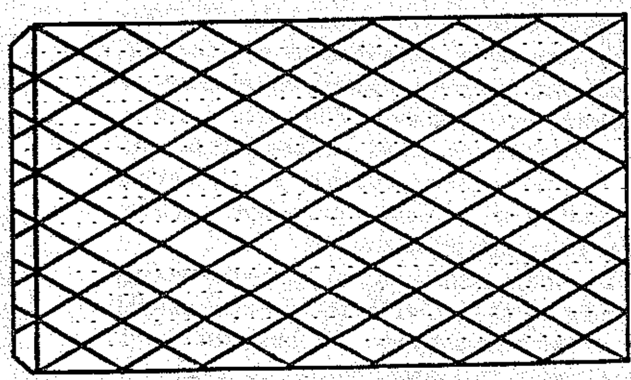


The sum of the areas are these mn triangles is

$$2mnR\sin\frac{\pi}{m}\sqrt{\frac{H^2}{n^2}+4R^2\sin^4\frac{\pi}{2m}}$$

To guarantee that the sides of all the triangles approach 0 take the limit as both m and n tend to infinity. If we let $m = n$ and then take the limit, we will get the desired area of $2\pi RH$. But if we first take $n = m^2$ we get $2\pi R\sqrt{1+R^2\pi^4/4}$. If first m tends to infinity and then n, the limit is infinity.[8]

To make a model of this polyhedral surface, take a piece of thin cardboard (a file folder works nicely) and mark it with a grid of diagonal lines as in the figure (the top of the figure is divided into m equal pieces, the side into n). Mark the back side of the cardboard with the dashed lines. Now lightly score each of the lines with a knife and fold back away from the lines. Next roll up the sheet into a cylinder, pressing to get the sheet to fold along the creases. Finally, glue the cylinder together using the flap.[9]



[8] These details are nicely worked out by Frieda Zames in "Surface area and the cylinder area paradox," *The Two-Year College Mathematics Journal*, 8 (1977), 207–211; reprinted in *A Century of Calculus, Part II, 1969–1991*, MAA, 1992, pp. 324–328 and also by Gail H. Atneosen, "The Schwarz paradox: An interesting problem for the first-year calculus student," Mathematics Teacher, vol. 65 (1972), pp. 281–284; reprinted in *Readings from the Mathematics Teacher. Calculus*, edited by Louise S. Grinstein and Brenda Michaels, NCTM, 1977, pp. 165–168.

[9] Dubrovsky, Vladimir, "In search of a definition of surface area. Now you see it, now you don't," *Quantum*, March/April 1991, pp. 6-9, and 44. The physical model of what they call Schwarz's boot, was designed by the Moscow architect and designer V. Gamayunov. It is described on p. 64.

IN PRAISE OF LIOUVILLE

Could this ever bee done
all problems whatever might bee resolved.
Isaac Newton, 1666[1]

There is no doubt that the availability of Computer Algebra Systems will change the way we teach mathematics. Soon our students will have symbolic integrators available in their palm-top computers. When the students start to use these computers experimentally and enter bizarre functions to integrate, they will quickly come across functions that their machines cannot handle. Then they will come to us, their teachers of a theoretical pencil and paper era, and ask the question we have always wanted to hear: Why? There will be many machine related reasons for the failure to integrate certain functions, but there is a theoretical reason that we must discuss: Some functions have no elementary antiderivatives.

The change of language in the previous sentence indicates that we will have to be more careful of our calculus terminology. We must distinguish between "integrals," "antiderivative,"and "elementary antiderivatives." To say that f has an antiderivative means that there is another function G, whose derivative is f. Now every continuous function f has an antiderivative, namely, $\int f(x)\,dx$. But this seems to be cheating. We would like to be able to integrate the function f using the integration techniques from a basic calculus course—to express the answer without the integral sign. In this case, G will be an elementary function, i.e., composed of compositions of alegbraic functions, roots, transcendental functions, and the inverses of such compositions. If we can do this, we say that f has an elementary antiderivative or that $\int f(x)\,dx$ is elementary.

From the time of Leibniz and Newton it was known that differentiation was an algorithmic process, whereas integration was not. It was not clear whether every function had an elementary antiderivative. Consequently, considerable effort was devoted to antidifferentiating wider and wider classes of functions. This topic was of great importance in the nineteenth century, and interest continues even today. As early as 1694 Jakob Bernoulli conjectured that

$$s = \int_0^r \frac{a^2}{\sqrt{a^4 - r^4}}\,dr$$

could not be integrated in elementary terms (this integral represents the length of a piece of the lemniscate $r^2 = a^2 \cos^2(\theta)$). Amazingly, Liouville proved that there are functions without antiderivatives.

Joseph Liouville (1809–1882) studied at the École Polytechnique and the École des Ponts et Chaussées, but he soon turned from engineering to research in mathematics and mathematical physics. Then he began a studious, but unexciting, fifty-year career as teacher at a variety of schools in Paris, interrupted only by his annual summer vacation and one brief sojurn into politics. Every student of complex analysis knows "Liouville's Theorem"[2] and many have used "Liouville's Journal" (*Journal des Mathématiques Pures et Appliquées*). In 1844 Liouville proved that transcendental numbers exist. He also made contributions to algebra, geometry, number therory, and celestial and rational mechanics.

In 1835 Liouville published a famous paper "Memoire sur l'integration d'une classe de fonctions transcendantes" (Memoir on the integration of a class of transcendental functions) in Crelle's Journal [1]. In this paper he provided necessary and sufficient conditions for the integrability in elementary terms of a rather large class of elementary functions. A special case of his result is:

Liouville's Theorem: If $\int f(x)e^{g(x)}\,dx$ is an elementary function, where f and g are rational

[1] So wrote Isaac Newton in his October 1666 Tract on Fluxions. To put 'this' in modern terms asks for an antiderivative for every function. See *The Mathematical Papers of Isaac Newton. Volume 1. 1664-1666*, edited by D. T. Whiteside, Cambridge University Press, 1967, p, 403.

[2] Every bounded entire function is constant. Alas, the theorem is due to Cauchy in 1844. The name arises because C. W. Borchardt heard Liouville lecture on the result in 1847 and attributed it to him.

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functions of x, and the degree of g is positive, then

$$\int f(x)e^{g(x)}\,dx = Re^{g(x)},$$

where R is a rational function of x.

If we assume this theorem without proof, then we can prove that certain integrals are not integrable in finite terms.

Example: $\int e^{-x^2}\,dx$ has no elementary antiderivative.

Proof: If we assume that e^{-x^2} has an elementary antiderivative, then, by Liouville's Theorem, it has the form

$$\int e^{-x^2}\,dx = Re^{-x^2},$$

where R is a rational function. Differentiating (and using the First Fundamental Theorem of Calculus) we obtain:

$$e^{-x^2} = R'e^{-x^2} + Re^{-x^2}(-2x).$$

Since the exponential function is never zero, we can cancel the exponential term and obtain $1 = R' - 2Rx$. If we replace the rational function R by P/Q, where P and Q are relatively prime polynomials and Q is not the zero polynomial, then, after some easy manipulations, we obtain:

$$Q(Q - P' + 2xP) = -PQ'. \qquad (\star)$$

Now if we assume that the degree of Q is positive, then by the Fundamental Theorem of Algebra (a result we assume already in high school, but never prove before a course in complex variables), we know that $Q(x) = 0$ has a root α of positive multiplicity r, i.e., $Q(x) = (x-\alpha)^r T(x)$, where $T(\alpha) \neq 0$. Since P and Q are relatively prime polynomials, $Q(\alpha) \neq 0$. We obtain a contradiction by observing that α is a zero of multiplicity at least r on the left-hand-side of $\star$, whereas α is a zero of multiplicity $r-1$ on the right (because differentiation reduces the multiplicity of a zero by 1).

This contradiction shows that Q is a non-zero constant polynomial. Thus $Q' = 0$ and so $\star$ becomes $Q - P' + 2xP = 0$. The polynomial P has some degree, say k, so P' has degree $k-1$ and $2xP$ has degree $k+1$. These three polynomials of different degree cannot add up to 0, so we have a contradiction. Q.E.D.

Recently a great advance was made on this work, one which is having a significant impact on computer algebra systems. It is

Risch's Theorem: There is an algorithm for deciding whether an arbitrary elementary function has an elementary antiderivative or not. Moreover, if a function has an elementary antiderivative, then the algorithm will find it. [2].

One of the real values of using a historical approach is that we can *talk* about mathematical ideas that are too hard to present in detail in class. The results are still important and of interest, even if the proofs cannot be given. Black holes, quarks, DNA, and plate tectonics are things that we have all heard about and understand in a general way, even though few of us know the technical details. This is a lesson that we had better learn from the physical scientists: Popular presentations of scientific ideas attract students to the field, and leaves the general public with warm feelings towards it.

We close with a short list of non-elementary integrals:

$$\int e^x/x\,dx \qquad \int dx/\ln(x)$$

$$\int \sin(x)/x\,dx \qquad \int \sin(x^2)\,dx$$

$$\int \sqrt{\sin(x)}\,dx \qquad \int \sqrt{1-x^3}\,dx$$

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[1] Liouville, Joseph, "Memoire sur l'integration d'une classe de fonctions transcendants," *Journal für die Reine und angewandte Mathematik*, 13(1835), 93–118.

[2] Mead, D. G., "Integration," *The American Mathematical Monthly*, 68(1961), 152–156.

[3] Risch, Robert, *Transactions of the American Mathematical Society*, 139(1969), 167–183.

[4] Risch, Robert, *Bulletin of the American Mathematical Society*, 76(1970), 605–608.

[5] Simmons, George F., *Calculus with Analytic Geometry*, New York: McGraw-Hill, 1985. See pp. 305–309.

[6] Taton, René, "Liouville, Joseph," *Dictionary of Scientific Biography*, 8, 381–387.

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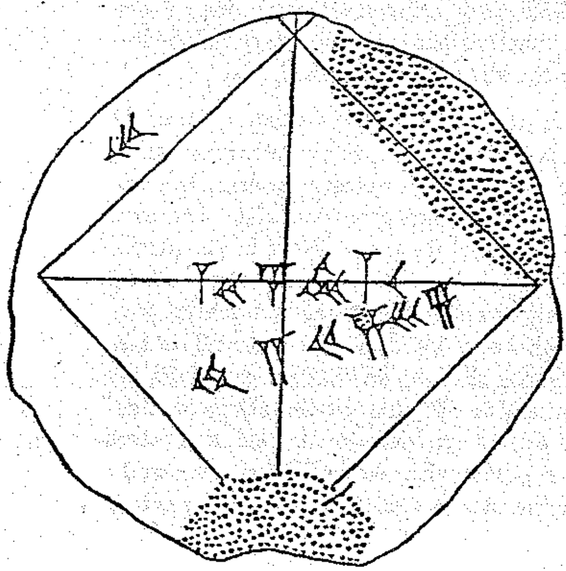
The First Sequence



YBC 7289, a tablet in the Yale Babylonian Collection, is pictured. It was written in cuneiform about 1800 B.C. and pictures a square, its diagonals, and three numbers expressed in sexagesimal notation. The numeral along the left side of the square expresses the number 30, the number along the diagonal is 1;24,51,10 and that below it is 42;25,35. The product of the first two of these is the third, and the second is a very good approximation to $\sqrt{2}$. This tablet also reveals that the Babylonians knew the Pythagorean Theorem (much stronger evidence for this is given by the tablet Plimpton 322).

Otto Neugebauer (1899–1990) and Abraham Sachs were the first to describe this tablet in their *Mathematical Cuneiform Texts*, which was published by the American Oriental Society and the American Schools of Oriental Research in 1945; see pp. 42–43. They conjectured—and no one since has doubted—that the Babylonians used the Divide and Average Method of computing square roots: First make a guess g_0 for the value $\sqrt{a}$. Then consider a/g_0. The product of these numbers is a, as desired. If the first is too small an approximation of $\sqrt{a}$, then the second is too large. Thus, if we average them we get an approximation which is better than either of the two. If we reiterate this process we obtain a convergent infinite sequence:

$$g_{n+1} = \frac{1}{2}\Big(g_n + \frac{a}{g_n}\Big). \qquad (\star)$$

Using this technique to approximate $\sqrt{2}$ in base ten[1], with an initial guess of 1, we obtain:

$$\begin{aligned}
g_0 &= 1\\
g_1 &= \frac{1}{2}(1+\frac{2}{1})\\
&= 1.5\\
g_2 &= \frac{1}{2}(1.5+\frac{2}{1.5}) = \frac{1}{2}(1.5+1.33\bar{3})\\
&= 1.416\bar{6}\\
g_3 &= \frac{1}{2}(1.416\bar{6}+1.41176470589\ldots)\\
&= 1.41421568628\ldots\\
g_4 &= \frac{1}{2}(1.41421568628\ldots+1.41421143847\ldots)\\
&= 1.41421356238\ldots
\end{aligned}$$

and $\sqrt{2} = 1.41421356237\ldots$

This is an amazingly close approximation, so it seems that the sequence $\{g_n\}$ converges to $\sqrt{2}$. Let us try to verify this.

First we use a technique that is very handy for finding limits of recursively defined sequences (I do not know the origin of this idea). Suppose (and this is a big supposition) that $\{g_n\}$ converges to l. If we take the limit on both sides of the relation $(\star)$ which defines our sequence, we obtain $l = \frac{1}{2}(l + \frac{a}{l})$. This quadratic has roots $l = \pm\sqrt{a}$, only the positive root of which is of interest to us. Thus, *if* the sequence converges, then it converges to $\sqrt{2}$.

[1] In base 60, if $g_0 = 1$, then $g_1 = 1;30$, $g_2 = 1;25$, and $g_3 = 1;24,51,10,\ldots$, precisely the value which appears on the tablet. This computation is simpler than in base ten for the reciprocal of g_1 was available in standard tables. However, that of g_2 "does not compute," i.e., does not have a terminating sexagesimal representation, but fortunately the Babylonians had an approximation for it.

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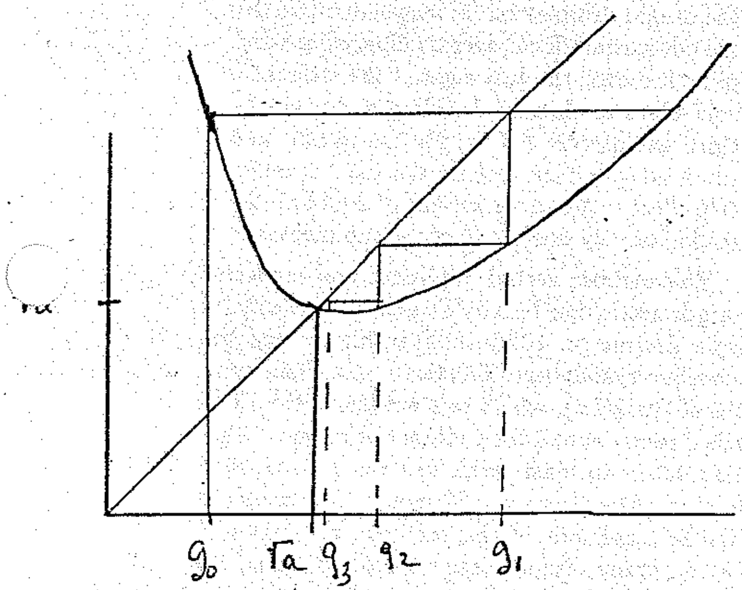
To show *that* the sequence converges, we mathematicians are likely to argue as follows: If g_0 is less than $\sqrt{a}$, then $g_1 > \sqrt{a}$ and the following terms form a decreasing sequence, so this bounded monotonic sequence converges. However, our students will be much happier if we draw a picture to illustrate the sequence $\{g_n\}$. Consider the function $g(x) = \frac{1}{2}(x + \frac{a}{x})$, which has, using the differential calculus, a minimum at $(\sqrt{a}, \sqrt{a})$. If the initial guess g_0 is less than $\sqrt{a}$, then $g(g_0) = g_1 > \sqrt{a}$. Now draw a horizontal line from the point $(g_0, g(g_0))$ to the line $x = y$ and then drop a perpendicular to the x-axis. This is the point g_1. Compute $g(g_1)$, draw a horizontal line to the diagonal $x = y$, and then drop down to g_2. Continue this process and generate the cobweb given in the figure. The geometry should readily convince students that the sequence converges.



The mathematical details of the previous paragraphs have been included to show that a historical approach need not be used throughout the problem. Here we only used the historical example to introduce a topic on infinite sequences, and then went ahead and proceeded as we ordinarily would. The historical beginnings show that the technique was important to mathematicians in the past. There is no conflict here, unless you leave the impression that the Babylonians did all of this.

The Divide and Average Root Method was, for centuries, attributed to Hero (or, less accurately, Heron) of Alexandria (fl. c. A.D. 62). See Sir. Thomas Heath's *A History of Greek Mathematics*, vol. 2, pp. 323–326 for additional information. The explanation of Neugebauer and Sachs provided a way of explaining how Archimedes obtained his square root approximations; previously this was a matter of considerable speculation.

When Newton's method is applied to $f(x) = x^2 - a = 0$ with first approximation x_1, the second is $x_2 = \frac{1}{2}(x_1 + \frac{a}{x_1})$. Thus the Divide and Average Square Root technique also follows from Newton's Method. The history presented above could also be used to motivate Newton's method. Of course, it would be silly to say that the Babylonians knew Newton's method.

A Second Sequence

Perhaps the most famous sequence of all time is that of Leonardo of Pisa, or less accurately, Fibonacci, which he introduced in 1202 in his *Liber abaci*.[2] In his *De nive sexangula* of 1611, Johannes Kepler (1571–1601), while writing about the divine proportion, saw a connection between it and the Fibonacci sequence. He wrote "It is impossible to provide a perfect example in round numbers. ... As 5 is to 8, so 8 is to 13, approximately, and as 8 to 13, so 13 is to 21."[3]

This passage is the origin of an important property of the Fibonacci sequence. This sequence is defined recursively by $F_0 = 1$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. We can turn Kepler's claim into an exercise which uses the above techniques by asking the student to find the limit of the sequence $\{F_{n+1}/F_n\}$.

[2] See D. J. Struik, *Source Book in Mathematics, 1200–1800*, pp. 1–4, for a translation of the text.

[3] Johannes Kepler, *The Six-Cornered Snowflake*, Oxford, 1966, p. 21. This is an English translation of Kepler's Latin pamphlet.

The Swineshead Series

If ever a man had an unfortunate surname, it was Richard Swineshead (fl.c. 1340–1355). It would be interesting to have a picture of him. He was one of the leading mathematicians and logicians at Merton College, Oxford, whose most important book, *Liber calculationum*, earned him the title of "The Calculator." This expository work, which shows the influence of the English Archbishop and mathematician Thomas Bradwardine (c. 1290–1349), became a standard reference work, dealing with every aspect of calculation and contained 49 rules dealing with movement. Here is the one that interests us most:

> If a point moves throughout the first half of a certain time interval with a constant velocity, throughout the next quarter of the interval at double the initial velocity, throughout the following eighth at triple the initial velocity, an so on ad infinitum; then the average velocity during the whole time interval will be double the initial velocity.

When presenting this as a problem in class, I leave out the the word "double" in the statement of the proposition, so that the students can figure that out for themselves. If the time interval and the initial velocity both have the value 1, this problem amounts to evaluating the infinite series $\sum_1^\infty n/2^n$. Except for the geometric series, instances of which were known in antiquity, this was the first infinite series which was summed.

The first time I presented this result was at the United States Military Academy at West Point NY in a class on the History of the Calculus. Thayer Hall, named after the father of the Academy, was under reconstruction (many years earlier it had been a riding arena) and there was an air hammer pounding in the next room. Those Army cadets never say die, so class continued. I had prepared a diagram and cutouts so that I could present this on the overhead projector. The small class was crowded around the overhead so they could hear me shout and when I put the cutouts on the overhead one cadet started moving the pieces around. Before I could finish my explanation of the problem, the cadet had provided the solution. This convinced me of the real value of tactile mathematics and I encourage you to make models and take them to class as often as possible.

Swineshead gave "a long an tedious verbal proof"[1] which is

> equivalent to arguing that the effect of doubling the velocity during the last half of the interval is equivalent to that of doubling it during the first half of the interval; the additional effect (over doubling) of tripling the velocity during the last quarter of the interval is equivalent to that of doubling it during the second subinterval (of length one-fourth); the additional effect (over tripling) of quadrupling it during the last eight of the interval is equivalent to that of doubling it during the third subinterval (of length one-eight); and so on ad infinitum. Hence the total cumulative effect is the same as that of doubling the initial velocity during all of the subintervals.[2]

This verbose verbal argument was converted to a geometric one by the French mathematician Nicole Oresme (c. 1320–1382) in his *De configurationibus qualitatum* (Treatise on the Configurations of Qualities), which was written in the 1350s while Oresme was at the College of Navarre. What I had taken to class with me was an overhead containing the diagram of Oresme. In accordance with the statement of the problem, the region is most naturally divided into vertical columns. To go with the diagram, I cut out several strips of paper. The first had width 1, the second width 1/2, the third width 1/4 , ... the n^{th} had width $1/2^n$. Each of them had height 1. To fit this nicely on an overhead the vertical scale should be about one-fourth the horizontal scale. These strips of paper are placed horizontally on the diagram, thereby "configuring" it in a different way. Then they are

[1] C. H. Edwards Jr., *The Historical Development of the Calculus*, New York: Springer, 1979, p. 91.

[2] Marshall Clagett, *The Science of Mechanics in the Middle Ages*, Madison: University of Wisconsin Press, 1959. This is an edition of the original documents, including a reproduction of the Oresme diagram.

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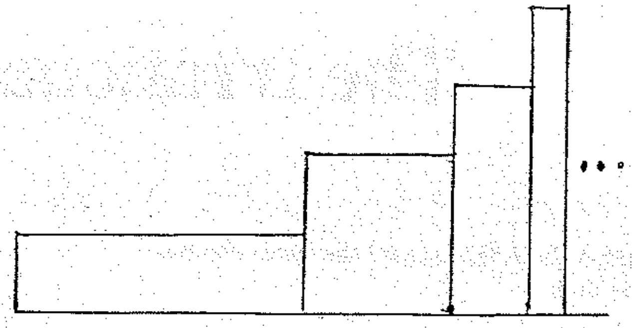
shifted left as in the diagram.

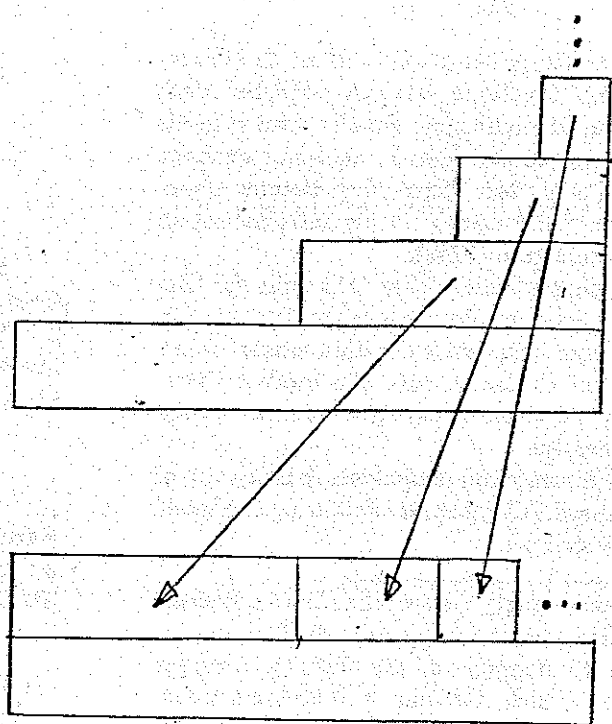


Arithmetically, the pieces in the vertical diagram represent the sum

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{n}{2^n}+\cdots$$

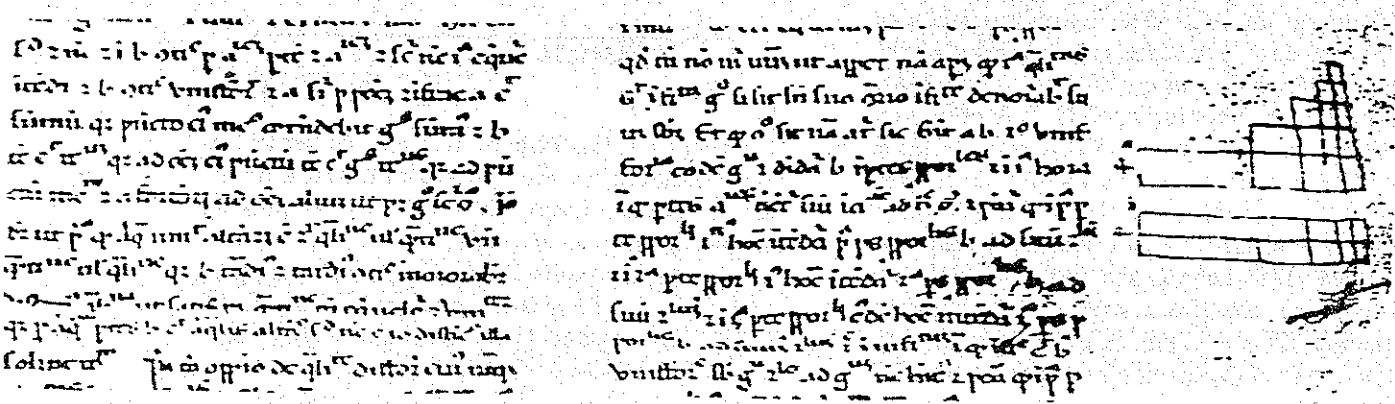
while those in the horizontal diagram represent

$$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots$$



What we have done is to reduce the Swineshead series to a geometric series. We should be sure to point out to our students that looking at the same thing various ways can be extremely fruitful; this is one of the primary ways that we get equations. We should not expect our students to absorb problem solving techniques like this all on their own—we need to reflect on what we do in class and to comment about the heuristic techniques which the students should emulate.

The importance of this work on infinite series, and that which continued in the fifteenth and sixteenth centuries, was not the particular results achieved but the change in mindset that the work engendered. The Greek horror of the infinite was gradually eroded during this period and mathematicians became more accepting of infinite processes in mathematics. This step was necessary before the calculus could come to fruition in the seventeenth century. This historical point is worth remembering when we teach calculus. Our students are not accustomed to thinking about infinite processes and so these processes should be approached slowly and carefully.



A page from a fourteenth-century manuscript illustrating the use of Oresme's configuration techniques to interpret text of Swineshead's *Calculationes*. MS Paris, BN lat. 9558, f. 6r.

The Irrationality of e

The twenty-year-old Euler used the letter e to designate the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

in a letter to Christiaan Goldbach on 25 November 1731, but the letter was not published until 1862 so it had no influence. He also used it in his *Mechanica sive motus scientia analytice exposita* (1736), vol. 2, p. 251. This usage became popular after the Euler used it in his *Introduction in analysin infinitorum* (1748).

Many people believe that 'e' stands for 'Euler,' but we have no documentation for this. In fact, to choose this letter for this reason seems wholly out of character with the modest Euler. More likely, he chose it as the first letter of the word exponential.

In his *Introduction in analysin infinitorum*[1] of 1748, Leonhard Euler gave the following continued fraction involving e:

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\ddots}}}}}}}$$

Euler used arithmetic to calculate this result from the numerical value that he knew for e, viz. $2,718281828459$ (yes, he used a decimal comma; but hereafter we shall use a decimal point). By a simple computation

$$\frac{e-1}{2}=0.8591409142295.$$

Euler then applied the Euclidean Algorithm, but we can use our calculator and get the same result. Take the reciprocal of the decimal on the right and then separate the integral part:

$$\frac{e-1}{2}=0.8591409142295=\frac{1}{1+.16395341374}$$

Again, take the reciprocal of the decimal and separate the integral part:

$$\frac{1}{1+.16395341374}=\cfrac{1}{1+\cfrac{1}{6+.09929355656}}$$

Continuing in this fashion, it is easy to verify Euler's continued fraction (except that round off error in your calculator will eventually accumulate to give incorrect quotients).

In his first paper on continued fraction (presented to the Petersburg Academy in March of 1737), Euler stated that although he had computed this continued fraction,

> I have only observed the arithmetic progression of the denominators and I have not been able to affirm anything except the probability of this progression continuing to infinity.

[1] This exceedingly rich work is now accessible in an English translation by John D. Blanton, *Introduction to Analysis of the Infinite,* Springer 1988, 1990. French, German, and Russian translations are also available. The Latin text is available as volumes 8 and 9 of series one of Euler's *Opera omnia*. A photographic reprint of the original was produced by Culture et Civilization, Bruxelles, Belgium, in 1967. The equation given is in §381 at the end of volume one. A copy of the original 1748 edition is on display at the MAA Headquarters in Washington D.C. It is often said that there are no diagrams in this work. This is only half true. There are no pictures in volume one, but the second, which deals with geometry, has 149. The MAA copy is interesting because, in accordance with a note to the binder which is still in the volume, the plates have been bound into volume one. This is handy in that the reader can then read from volume two while looking at plates in volume one.

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Therefore, I have exerted myself in this above all: that I might inquire into the necessity of this progression and prove it rigorously.[2]

Euler was so concerned about proving this result rigorously because he had made the simple observation in this paper that a continued fraction terminates iff it represents a rational number. Thus if he could prove that the denominators of the above continued fraction are an arithmetic progression then the continued fraction would be infinite and $(e-1)/2$ would be irrational. From this it would follow elementarily that e itself was irrational.

Euler *knew* that e was irrational, but he was unable to prove it. This is the earliest distinction between truth and proof that I am aware of. That Euler admitted that he could not prove the result is a result of his outstanding character and interest in advancing mathematics.

The irrationality of e was first proved by Euler's colleague at the Berlin Academy Johann Heinrich Lambert (1728–1777) in 1761 and published in 1767.[3] Lambert proved the more general result that if x is a non-zero rational, then e^x is irrational. His proofs, however, are too complicated to be used in class.

The elementary proof of the irrationality of the number e which commonly appears in textbooks is due to J. B. Fourier (1768–1830) about 1815:[4]

Suppose that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

is a rational number p/q. Multiply both sides of this equation by $q!$ and transpose all integral terms to the left hand side of the equation:

$$\begin{aligned} q!\,(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{q!}) & \\ &= q!\,(\frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \cdots) \\ &= \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \cdots \\ &< \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \cdots \\ &= \frac{1}{1-(1/(q+1))} - 1 \\ &= 1/q \\ &< 1 \end{aligned}$$

The contradiction here is that the first expression is a positive integer, and it has been proved to be less than one.

It was not until 1873 that Charles Hermite (1822–1901) showed that e is transcendental, i.e., not the root of any polynomial equation with integral coefficients.[5]

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[2] "De fractionibus continuis dissertatio," *Commentarii Academiae Scientiarum Petropolitanae*, 9 (for 1737, published 1744), pp. 98–137 = *Opera omnia*, series I, volume 14, 187–215. There is an English translation by Bostwick F. Wyman and Myra F. Wyman, "An essay on continued fractions," *Mathematical Systems Theory*, vol. 18 (1985), 295–328.

[3] "Vorläufige Kenntnisse für die so die Quadratur und Rectification des Cirkuls suchen" (Preliminary knowledge for those who seek the quadrature and rectification of the circle), *Beiträge zum Gebrauche der Mathematik und deren Anwendung II*, pp. 140–169.

[4] According to the analyst Ernest William Hobson (1856–1933) in his *"Squaring the Circle"; A History of the Problem* (1913; reprinted Chelsea 1953 in *Squaring the Circle, and Other Monographs*, especially p. 44) this result of Fourier's was first published in *Mélanges d'analyse algébrique et de géométrie* (Paris: Ve Courcier, 1815), but I have never seen this work. A slightly different proof is in Konrad Knopp, *Theory and Application of Infinite Series* (1951), pp. 194–195. Knopp often gives interesting historical remarks and references to the original papers, but there is none on this topic.

[5] Hermite's original paper is "Sur la fonction exponentielle," *Comptes Rendus*, vol. 77, pp. 18–24, 74–79, 226–238, 285–292 or *Œuvres de Ch. Hermite*, volume 3 (1912), pp. 150ff. For more on the history of e, see the following: Eli Moar, *e. The Story of a Number*, Princeton 1994. J. L. Coolidge, "The number e," *American Mathematical Monthly*, 57(1950), 591–602. U. G. Mitchell and Mary Strain, "The number e," *Osiris*, 1(1936), 476–496.

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History of the Brachistochrone

In April 1696, Isaac Newton, then 53, resigned the Lucasian Professorship at Cambridge to devote the remaining three decades of his life to "ye Kings business" at the Mint in London.[1] Just nine months later, on January 29, 1696/7[2], Newton received a letter from France, probably from Varignon, but possibly L'Hospital, containing two challenge problems. The fly-sheet on which this "Programma" was printed at Groningen was dated 1 January 1697 and addressed

> To the sharpest mathematicians now flourishing throughout the world, greetings from Johann Bernoulli, Professor of Mathematics.

Bernoulli's stated aims in proposing the problem are admirable:

> We are well assured that there is scarcely anything more calculated to rouse noble minds to attempt work conductive to the increase of knowledge than the setting of problems at once difficult and useful, by the solving of which they may attain to personal fame as it were by a specially unique way, and raise for themselves enduring monuments with posterity. For this reason, I ... propose to the most eminent analysts of this age, some problem, by means of which, as though by a touchstone, they might test their own methods, apply their powers, and share with me anything they discovered, in order that each might thereupon receive his due meed of credit when I publically announced the fact. [Scott 1967a, p. 224]

Bernoulli's new year's present to the mathematical world was the problem:

> To determine the curved line joining two given points, situated at different distances from the horizontal and not in the same vertical line, along which a mobile body, running down by its own weight and starting to move from the upper point, will descend most quickly to the lowest point. [Scott 1967a, p. 225]

Johann Bernoulli christened this enigma the brachistochrone problem, a word he coined from the Greek words 'brachistos' meaning shortest and 'chronos' meaning time. He added that the solution was not a straight line, but a curve well known to geometers. [At this point I am careful not to reveal the solution to the students.]

The problem was not new. In 1638 Galileo attacked it in his last work *Discourses and Mathematical Demonstrations Concerning Two New Sciences*, generally known in English by the last words of the title. He was only able to prove a circular arc was better than a straight line of descent, although he incorrectly concluded that a circular arc was the solution.

In fact, Bernoulli had stated the problem earlier. How he happened to offer it to the world in 1697 as a New Year's Day present is an interesting part of the story. In the *Acta eruditorum* of June 1696 (pp. 264–269), Bernoulli had attempted to show that the calculus was necessary and sufficient to fill the gaps in classical geometry. At the very end of the paper he tacked on the brachistochrone problem as a challenge. When the six months allotted for the solution were up, Bernoulli had received no correct solutions. He had received a letter from Leibniz praising the problem and indicating that he had solved it in one evening. Indeed he had found the differential equation describing the curve, but he had not yet recognized the curve as an inverted cycloid. Bernoulli and Leibniz interpreted Newton's six month silence to mean the problem had baffled him—indeed he had not seen it. Thus they intended to demonstrate the supe-

[1] I have used this material at the end of a two or three semester calculus sequence. My purpose for presenting it there, rather than in a differential equations course, is that it pulls together many topics from the calculus. It is also a tour de force that presents a beautiful piece of mathematics. On several occasions I have covered the mathematics and history in one fifty minute class period. The historical details given here are somewhat more detailed than I actually present, as they are intended for instructors rather than students. In particular, I try to debunk the myths common in this oft-told tale.

[2] The slash date indicates that the Gregorian calendar was not yet in use in England; the date on the continent was February 8, 1697

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riority of their methods publicly [remember, the priority dispute was just beginning]. Thus Leibniz suggested the deadline be extended to Easter and that it be distributed more widely. So Bernoulli added a second problem[3], had a broadside published, and made sure it circulated widely. This was his New Year's "Programma."

The brachistochrone problem was a difficult one. In France, Pierre Varignon admitted that he was "immediately rebuffed by its difficulty," and L'Hospital pleaded that it would need to be "reduced to pure mathematics" before he could attempt it, "for physics embarrasses me." In Oxford, John Wallis was stumped and David Gregory wasted two months trying to prove that the catenary was the desired curve before Newton set him right.

Thirty years later Newton's niece Catherine Barton Conduitt recalled,

> When the problem in 1697 was sent by Bernoulli— Sr. I. N. was in the midst of the hurry of the great recoinage [and] did not come home till four from the Tower very much tired, but did not sleep till he had solved it wch was by 4 in the morning. [Westfall 1980a, 582–3; Whiteside 1981a, 72–73.]

Although she probably heard this story from Newton later, rather than being a witness herself for she was probably not yet his housekeeper [See Westfall 1980a, p. 595], there is no reason to doubt it. Indeed, the next day Newton sent his answer to his old Cambridge friend Charles Montague, who was then President of the Royal Society. He did not send any justification that the answer was a cycloid [see Scott 1967a, p. 226, or Whiteside 1981a, p. 75, where his "solution" takes but one paragraph]. The original worksheets are not extant [Whiteside 1981a, p. 74], which is surprising given Newton's propensity to save every scrap of paper he ever wrote on. Newton did not instruct Montague on what to do with the answer. Even though the "Programma" explicitly said they should be sent to Bernoulli, Montague immediately had the answer published anonymously in February in the *Transactions of the Royal Society* [vol. 17, no. 224 (for January 1696/7), pp. 384–389]. Thus the trap that Bernoulli and Leibniz had set for Newton failed to snare its game.

Derek T. Whiteside, who has published an extremely valuable edition of *The Mathematical Papers of Isaac Newton* in eight volumes (with an index volume yet to follow), claims that the fact that it took Newton twelve hours to solve these problems indicates that his mathematics was rusty from nine months disuse. It also shows that the gradual decline[4] in Newton's mathematical ability had set in. However, his solution of the brachistochrone problem is one piece of counterevidence to the myth that Newton's old age was mathematically barren [Whiteside, 1981a, pp. xii, 3].

Immediately on receiving the solution of the anonymous Englishman via Basnage de Beauval, Bernoulli wrote Leibniz that he was "firmly confident" that the author was Newton. Leibniz was more cautious, admitting only that the solution was suspiciously Newtonian. Several months later Bernoulli wrote de Beauval that "we know indubitably that the author is the celebrated Mr. Newton; and, besides, it were enough to understand so by this sample, ex ungue Leonem." Within a few weeks this shrewd guess was common knowledge across Europe. Unfortunately the phrase, "from the paw of the lion," was so scrambled (an initial "tamquam" was added) by Newton's biographers Woodhouse and Brewster that it has "travelled from a mere pedestrian *cliché* to be a universally parroted (but no less spurious) myth" [Whiteside 1981a, pp. 9–10]. The phrase goes back to Plutarch and Lucian, who allude to the sculptor Phidias' ability to determine the size of a lion given only its severed paw. In Bernoulli's day the phrase did not carry the power it does today.

Not having succeeded in trapping Newton, Bernoulli quickly lost interest. Thus it was left to Leibniz to publish in the May 1697, *Acta* (pp. 201–224) the solutions received by the deadline. These included Johann's own solution *"Curvatura radii in diaphanis non uniformibus ..."* [The curvature of a ray in nonuniform media], one by his older brother Jakob *"Solutio problematum fraternorum ..."* [Solution of a problem of my brother], one by L'Hospital (probably not completely indepen-

[3] Bernoulli's second New Year's Day problem was: To find a curve such that the sum of the two segments PK and PL, on a line drawn at random from a point P to cut the curve in two points K and L, though the two segments be raised to any power, is a constant.

[4] The brachistochrone problem should not be confused with Bernoulli's challenge problem of December 1715, asking for the family of curves orthogonal to a given family, which was designed to test the "pulse of the English analysts." Newton fared much worse here. See Whiteside 1981a, p. 504.

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dently), one by Tschirnhaus, and a reprint—seven lines in all—of Newton's. This time Newton was not anonymous, for Leibniz had mentioned him in his introductory note (p. 204). Leibniz was so embarrassed by the whole thing that he wrote the Royal Society indicating that he was not the author of the challenge problem. Technically, this was true, but he had contrived with Bernoulli to embarrass Newton.

Within a few years there were solutions by John Craige, David Gregory, and Richard Sault. In 1699 Fatio de Duillier published a solution, though the paper is better remembered for naming Leibniz as "second inventor" of the calculus. By 1704 Charles Hayes, in his widely read *Treatise on Fluxions*, presented it as a mere worked example in a textbook. As often happens, a difficult problem, once cleverly solved, comes within the grasp of many.

Johann's solution[5] was elementary and clever: the path of quickest descent is the same as a light ray passing through a fluid of variable density. This is the solution we will present. It is important because it presents the first example of the optical-mechanical analogy. See Struik 1969a for an English translation and Simmons 1972a for a modern presentation of the mathematics. Jakob's solution was geometrical and laborious. But it was more general and an important early step in the calculus of variations. He realized that this was a new *type* of problem—the variable is a function.

Newton did not think kindly of Bernoulli's challenge for he wrote Flamsteed two years later, "I do not love to be printed upon every occasion much less to be dunned and teezed by forreigners about Mathematical things or to be thought by our own people to be *trifling* away my time about them when I should be about ye Kings business" [Scott 1967a, p. 296]

The Mathematics

[5] Carl Boyer, *A History of Mathematics*, p. 457, indicates that Johannes Bernoulli, "found an incorrect proof that the curve is a cycloid, and he challenged his brother to discover the required curve. After Jacques [Jakob] correctly proved the curve sought is a cycloid, Jean [Johannes] tried to substitute his brother's proof for his own." While this story certainly fits in with the character of the family, I know of no justification for it. Might this relate to the catenary? See Kline, *Mathematical Thought form Ancient to Modern Times*, pp. 472–473.

Imagine a curved wire from A to B and a bead sliding down the wire with friction neglected. Which curve minimizes the descent time? Galileo proved a circular path was better than a straight line path. But that is not the minimum. Here is how Johann Bernoulli solved the brachistochrone problem in 1697.

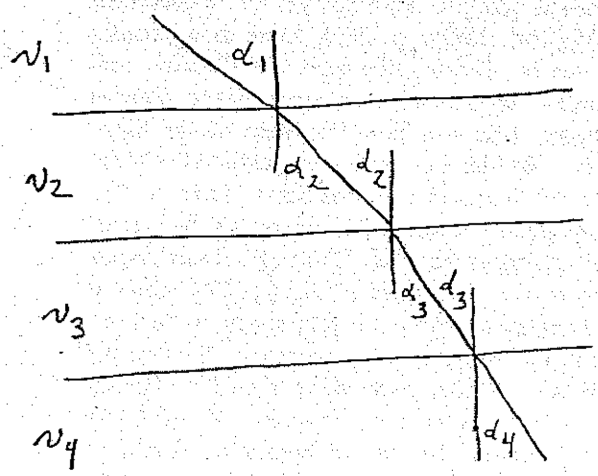
Some Optics: In the differential calculus we considered the behavior of a ray of light as it passed from air into water, and derived Snel's law of optics:

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}$$

where α_1 is the angle of incidence, α_2 the angle of reflection, and v_1 the velocity of light in air, and v_2 its velocity in water.

Now if we consider what happens to light when it passes through a medium of variable density. Suppose we have a stratified optical medium, with the velocity constant in the individual layers. Then we obtain

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} = \frac{\sin\alpha_3}{v_3} = \cdots$$



Now if the layers grow thinner and more numerous, we obtain, in the limit,

$$\frac{\sin\alpha}{v} = k \qquad (1)$$

where k is constant.

Some Mechanics: A freely falling body has constant accelleration, i.e.,

$$\frac{d^2y}{dt^2} = g.$$

Integrating, we obtain

$$v = dy/dt = gt + v_0, \qquad \text{where} \quad v(0) = v_o$$

and

$$y = \frac{1}{2}gt^2 + v_0t + y_0, \qquad \text{where} \quad y(0) = y_o.$$

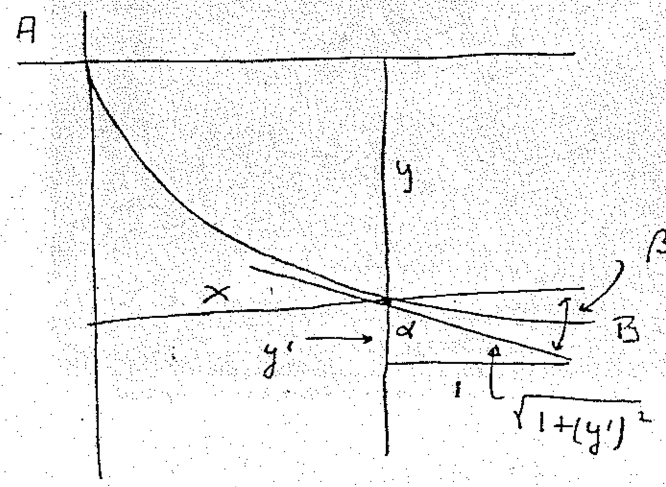
$v = gt$

If the body falls from rest starting at $y = 0$, then $y = gt$ and $y = \frac{1}{2}gt^2$. Eliminating t, we obtain the useful equation

$$v = \sqrt{2gy}, \tag{2}$$

which gives velocity in terms of the distance fallen.

Some Calculus: From the diagram



one can read off

$$\sin\alpha = \frac{1}{\sqrt{1+(y')^2}} \tag{3}$$

or, if some student insists, it can be derived:

$$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(y')^2}}.$$

Combining (1), (2), and (3) we obtain a differential equation describing the motion of the body sliding down curve:

$$k = \text{constant} = \frac{\sin\alpha}{v} = \frac{1}{\sqrt{2gy}\sqrt{1+(y')^2)}}$$

or

$$y(1+(y')^2) = c, \qquad \text{where} \quad c = 1/2gk^2.$$

To solve this equation, separate the variables and integrate:

$$\int dx = \int \sqrt{\frac{y}{c-y}}\, dy.$$

To evaluate the second integral, use the unusual substitution

$$\tan\theta = \sqrt{\frac{y}{c-y}}$$

which is, after some trigonometry, $y = c\sin^2\theta$. Thus

$$\begin{aligned} x &= \int \sqrt{\frac{y}{c-y}}\, dy \\ &= \int \tan\theta \cdot 2c\sin\theta\cos\theta\, d\theta \\ &= c\int 2\sin^2\theta\, d\theta \\ &= c\int (1 - \frac{1}{2}\cos\theta)\, d\theta \\ &= \frac{c}{2}(2\theta - \sin(2\theta)) + c_1. \end{aligned}$$

If we choose our coordinate system so that A is at the origin, then $x(0) = 0$ and $y(0) = 0$. So $\tan\theta = 0$ and $\cancel{\theta = 0}$. Consequently, if $x = y = \theta = 0$, we have $c_1 = 0$. This yields the parametric form of the solution:

$$x = \frac{c}{2}(2\theta - \sin(2\theta))$$

$$y = \frac{c}{2}(1 - \cos(2\theta)),$$

where the last is obtained by a trigonometric identity on the substitution we made. These are the parametric equations of a cycloid where the radius of the generating circle is $c/2$ and θ is ?????.

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Hall, A. Rupert

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Westfall, Richard S.
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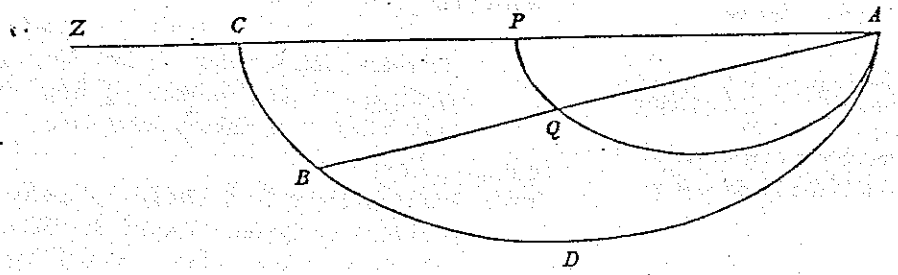
Whiteside, D. T.
1981a *The Mathematical Papers of Isaac Newton, Volume VIII, 1697–1722*, Cambridge University Press. Contains excellent and very scholarly commentary by Whiteside (pp. 1–14) as well as the pertinent documents (72–90) and numerous references. This is the primary source of my information.

Biographical information of high quality on all of the individuals mentioned above can be found in the *Dictionary of Scientific Biography*, edited by C. C. Gillispie.

Investiganda est curva Linea ADB in qua grave a dato quovis puncto A ad datum quodvis punctum B vi(6) *gravitatis suæ citissimè descendet.*



Cauchy's Famous Wrong Proof

The following example stems from my own teaching experience. Once, when I came to the topic of sequences and series of functions while teaching an undergraduate analysis class, I realized that the book had done a particularily poor job on this topic (to protect the guilty, no reference will be given). Thus something had to be done. Since I had been rereading Imre Lakatos's delightful little book, *Proofs and Refutations*, I decided to see if his analysis of Cauchy's famous wrong proof could be adapted to the classroom.[1]

In presenting the topic of sequences and series of functions, I began, as always, with a goodly supply of carefully chosen examples, drew pictures of some of them, and left others for homework. After noting how nicely the examples behaved, I coaxed the following observation out of the students.

THEOREM. A convergent series of continuous functions converges to a continuous function.

After congratulating my students for making this brilliant conjecture, I pulled Cauchy's *Cours d'Analyse* (1821) from my briefcase, attributed the theorem to him, and then presented his proof. This dusty tome lent authority to the argument to be given below, and the students were pleased that I had taken the trouble to go back to original sources. As often as appropriate, I take relevant books to class, and after explaining how they bear on the material, I pass them around for the students to look at. If that is impossible, I try to have excerpts on overhead transparencies to show. The students appreciate this.

Before giving the proof, some notation needs to be introduced. Cauchy is dealing with a series of functions whose sum is the function s. The n^{th} partial sum of this series is denoted s_n, and the remainder by r_n. Today all of this is usually presented in terms of sequences, but I wanted to follow Cauchy fairly closely. Consequently, I read Cauchy's proof to them and wrote it on the board in translation:

> When the terms of the series contain the same variable x, and this series is convergent, and its different terms are continuous functions of x, in the neighborhood of a particular value assigned to this variable; and s_n, r_n and s are again three functions of the variable x, of which the first is evidently continuous with respect to x in the neighborhood of the particular value in question. This assumed, let us consider the increases that these three functions receive when one increases x by an infinitely small quantity. The increase of s_n will be, for all possible values of n, an infinitely small quantity; and that of r_n will become insensible at the same time as r_n [sic], if one assigns to n a very considerable value. Hence, the increase of the function s can only be an infinitely small quantity. From this remark, one deduces immediately the following proposition [i.e., the Theorem above].[2]

[1] Imre Lakatos, *Proofs and Refutations*, edited by John Worrall and Elie Zahar, Cambridge University Press, 1976, pp. 123–141.

[2] Augustin Cauchy, *Cours d'Analyse*, 1821, p. 120. Reprinted in his *Œuvres*, II, 3.

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Since the students were familiar with Weierstrassian ϵ-δ techniques, I took the time to carefully formulate Cauchy's proof in the modern language that they were learning to understand.

PROOF: Let $\epsilon > 0$ be given. Then
(1) Since each function is continuous, their partial sum, $s_n(x)$, is continuous,

$$\exists\delta\,\forall a \quad |a| < \delta \Rightarrow |s_n(x+a) - s_n(x)| < \epsilon.$$

(2) Since the series converges at x,

$$\exists N\,\forall n > N \quad |r_n(x)| < \epsilon.$$

(3) Since the series converges at $x+a$,

$$\exists N\,\forall n > N \quad |r_n(x+a)| < \epsilon.$$

Thus:

$$\begin{aligned}|s(x+a) - s(x)| &\\ &= |s_n(x+a) + r_n(x+a) - s_n(x) - r_n(x)|\\ &\leq |s_n(x+a) - s_n(x)| + |r_n(x)| + |r_n(x+a)|\\ &\leq 3\,\epsilon\end{aligned}$$

Hence the function s is continuous. [It is pedantic to insist on ending with precisely "ϵ." Why make the mathematics even more mysterious for the student.] Q.E.D.

By careful planning, the class ended just as the proof did, and I was relieved that there were no questions after class. The next day the students were upset, for they had done their homework (this was a good class) and observed that some of the examples I had given (and left the graphs as exercises) contradicted Cauchy's Theorem. But they were ready to do mathematics.

I asked about the counterexample they had discovered in their homework:

$$\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \cdots$$

Were all the terms of the series continuous functions? Did the series converge? Was the limit function really discontinuous? "Yes, yes, yes." they said. Well then, what about the theorem? Cauchy published it in his *Cours d'Analyse*, so it must be correct, right? "Yes," they readily agreed. They had also accepted the proof when it was presented in class, for it seemed correct to them. They were puzzled. Something was wrong, but what?

I asked if they had examined the proof to see if anything was wrong with it. No, that had not occurred to them. So I suggested that we should look at the proof carefully.

Imre Lakatos makes the argument that it was in the mid-nineteenth century that mathematicians made the same advance as my students were now making: When a proof is wrong, do not just abandon it, but analyze it carefully to see if there are any "hidden hypotheses" that would make it correct. Lakatos took this phrase from a student of Dirichlet, Philipp Ludwig von Seidel (1821–1896), who used it in 1847 when he took the steps that my students were now ready for.

Thus, we shall now analyze Cauchy's proof. In step (1) above we need to realize that δ depends on ϵ, x, and n. To make this explicit, we shall write $\delta(\epsilon, x, n)$. Now, in step (2), N depends on ϵ and x, so we write $N(\epsilon, x)$. However, in step (3), N depends on ϵ, x, and ALSO on a. Using the same notation, we express this by $N(\epsilon, x+a)$. Now comes the critical observation. To make Cauchy's proof work, we need an integer M bigger than $N(\epsilon, x)$ and simultaneously bigger than $N(\epsilon, x+a)$ for each a whose absolute value is less than $\delta(\epsilon, x, n)$. Thus we must know that

$$M = \text{Max}_t N(\epsilon, t)$$

exists for all ϵ, i.e., that M does not depend on x. Consequently, the additional hypothesis that we need is the following:

$$\forall\epsilon > 0\ \exists M\ \forall n > M\ \forall x \quad |r_n(x)| < \epsilon.$$

This is the definition of uniform convergence, and is precisely what is needed to make Cauchy's theorem correct and the proof work.

What I have done here is to motivate the definition of uniform convergence. Had I just written it down in the usual definition-theorem-proof style of modern mathematics, it would appear to be very much ad hoc. The historical presentation allows the student to see the true origin of the concept. As Lakatos has observed, the correct concept is generated by the incorrect proof. *This is one case where I feel that a historical presentation is absolutely necessary to the understanding of the material.*

You may object that this type of presentation takes too much time, for it did take two whole class periods. But that is not so. The time was well spent. Presenting the wrong proof and then

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analyzing it to see what additional hypotheses are needed takes far less time than presenting an ad hoc definition, trying (probably unsuccessfully) to explain it, and then finally giving the proof. With my presentation there is no need to give a correct proof after the definition has been discovered; that is an easy exercise for the student. In fact, my students said there was no reason for me to write out a new proof, for they had a deep understanding of how it works. Moreover, with this presentation the students have also learned more. The opportunity to analyze an incorrect proof builds both confidence and skepticism (students must learn that books may contain errors). More importantly, it shows them where theorems come from: We make conjectures, attempt proofs, analyze them, and refine them. It also shows the importance of definitions, showing that they are carefully chosen, not things arbitrarily written down just before a proof. I trust you will agree with my assessment that without giving this historical presentation, the student's understanding of the concept of uniform convergence would be severely hampered.

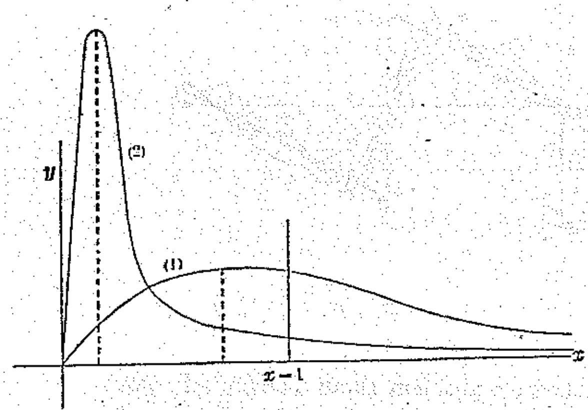
In this example, the history has stayed in the background, but by the time I had finished, the students were anxious to have some details. Since this whole issue has been extensively and hotly debated in the literature over many years, I shall refrain from giving the historical details here. Many of them are in Lakatos's book. For a current entry into the literature, see Laugwitz.[3] Nonetheless, I must end with a historical point. In 1826, a mathematician wrote "... it seems to me that the theorem admits of exceptions" and then provided the first counterexample, the same counterexample that my students had done for homework. The mathematician was the Norwegian, Niels Henrik Abel (1802–1829).[4]

Pictures

The presentation of this material is greatly enhanced today by the use of graphers. The first to adopt the graphical approach was William Fogg Osgood (1864–1943), who, after receiving his Ph.D. under the influence of Felix Klein at Erlangen in1890 taught at Harvard from 1890 to 1933.[5] The first example that Osgood considers is the sequence of functions

$$s_n = nxe^{-nx^2}$$

where to obtain the graph of the general curve (2), "it is sufficient to divide the abscissas and multiply the ordinates of (1) by $\sqrt{n}$."[6]



This sequence of functions is not uniformly convergent, as is immediately evident from the diagrams.

> For draw the curves $y = f(x)+\epsilon$, $y-f(x)-\epsilon$. Then it is clear that m cannot be taken so large that the approximation curve $y = s_n(x)$ will lie wholly within the belt thus marked off.[7]

[3] Detlef Laugwitz, "Infinitely small quantities in Cauchy's textbooks," *Historia Mathematica*, 14 (1987) 258–274.

[4] Niels Henrik Abel, "Untersuchungen über die Reihe: $1+\frac{m}{1}x+\frac{m\cdot(m-1)}{1\cdot 2}x^2+\frac{m\cdot(m-1)\cdot(m-2)}{1\cdot 2\cdot 3}x^3+\cdots$ u. s. w.," *Journal für die reine und angewandte Mathematik*, 1(1826), 311–339. Reprinted in French translation in Abel's *Œuvres complètes* (1881), 1, 219–250. Also in Ostwald's *Klassiker*, #71.

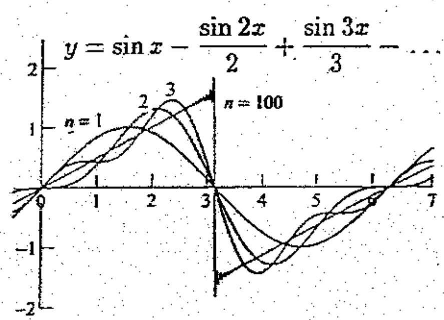
[5] For information on Osgood see J. L. Walsh, "William Fogg Osgood," pp. 79–85 in *A Century of Mathematics in America, Part II*, edited by Peter Duren et al., AMS, 1989 and *A Semicentennial History of the American Mathematical Society, 1888–1938* by R. C. Archibald, AMS 1938, pp. 153–158, which contains a bibliography of his works.

[6] W. F. Osgood, "A geometrical method for the treatment of uniform convergence and certain double limits," *Bulletin of the American Mathematical Society*, series 2, volume 3 (1897), pp. 59–86.

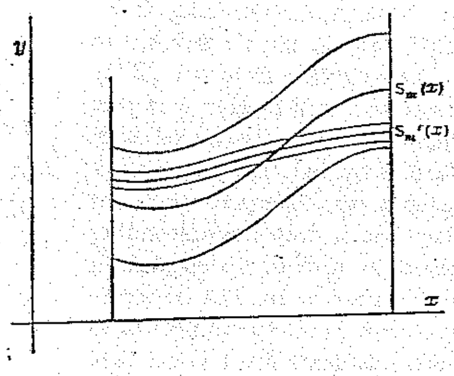
[7] Osgood, *op cit.*, p. 66. Matthias Kawski of Arizona State University pointed out on the calc-reform email list (24 May 1996) that there is a

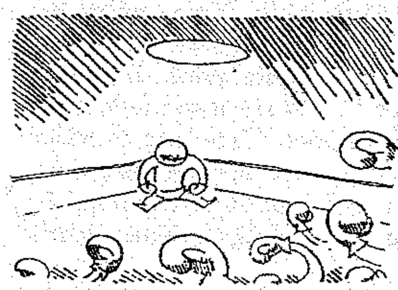
Exercises:

1. Work out the details of Abel's counterexample to Cauchy's "Theorem." This is a fairly hard example to work out in detail, but it is easy to convince yourself by the use of a grapher:[8]



2. Construct a problem about Cauchy's Famous Wrong Theorem that shows the Max can be infinite.[9]





PROBLEMS

Problems worthy of attack prove their worth by hitting back.

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beautiful graphing calculator view of the difference between uniform continuity and pointwise continuity (and which is applicable, mutatis mutandis, to the corresponding notions of convergence). Pictorically, a function is continuous at a point if given a viewing window of any height (2ϵ) centered at that point, one can always find a width (2δ) such that the graph exits the window only through the sides and not through the top and bottom. This is nice for classroom exploration because students will naturally choose different heights for their windows. Now if one traces the graph, keeping the same window size, and if the graph never exits the top or bottom of the window, then the function is uniformly continouous over the interval that you have traced. Students easily observe that continuity implies uniform continuity on closed bounded intervals.

[8] This picture is from E. Hairer and G. Wanner, *Analysis by Its History*, Springer, 1996, p. 212. This book contains a great deal of interesting historical material.

[9] An earlier version of this note appears in my paper "My favorite ways of using history in teaching calculus," pp. 123–134 in *Learn From the Masters*, edited by Frank Swetz et alia, MAA, 1995.

Preparing Overhead Transparencies

> The most frequent compliant heard at meetings is that transparencies are unreadable.[1]

A wonderful way to enrich presentations on the history of mathematics is to include pictures of the mathematicians involved, maps of where they worked, title pages of the books being discussed, diagrams used in their works, and selected pages showing important mathematical developments. In addition to these illustrations, there are ample opportunities to use overheads of a purely textual nature to support your spoken message. For example, a chronology of the individuals life, an especially pithy quotation, a definition or theorem that you wish to discuss carefully, an important reference, etc. Without much effort you can think of many things which would spice up your presentation.

Unfortunately, moving from creative teaching ideas to effective implementation requires a considerable investment of time and effort. How many times have you attended the lecture where the speaker has photocopied a page from a book and made an overhead transparency from it? No one more than 5 meters from the screen can read it. What's worse, the speaker often knows it and apologizes.[1] My purpose here is to help you avoid some of the pitfalls of a poor presentation.

[1] Robert M. Fossum and Kenneth A. Ross, *Presentation of Mathematical Papers at Joint Mathematics Meetings and Mathfests*, AMS and MAA, n.d.

[1] See "You won't be able to read this, but ... " by Charles E. Engel, *International Science Review*, ???—NEED EXACT REFERENCE—???. He cites C. J. Duncan, *Modern Lecture Theatres*, London: Oriel Press, 1966, to support the recommendation that "the width of the projected image is not less than one-sixth of the distance from the screen to the most distant viewer." Alas, the speaker has no control over the size of the room, and little over the projection equipment, so it is wise to heed the advice that "not more than ten lines on a transparency or slide will conform with the requirements for good visual acuity."

Finding Interesting Illustrations

The hardest part of preparing good overheads is in finding and selecting the visuals that you want to use. This requires a lot of effort and diligence over the long term. But it is well worth it. Carefully prepared illustrations can make a good presentation into a superb one. You will find that good overheads can be used in a variety of situations, so it is useful to build up a substantial collection of overhead for use in class and in public presentations.

Some periodicals, such as *The Mathematical Intelligencer*, contain many excellent photographs of mathematicians, places where they lived and worked, and illustrations from their works. However, photographs can be very hard to reproduce as overhead transparencies because of the lack of contrast — the various shades of gray all come out black, and so the end product is often unacceptable. Of course transparencies for use in a slide projector can be made photographically, but that is more expensive, requires special equipment during your presentation, and is less convenient both in preparation and use. Thus I shall restrict my attention here to the use of the overhead.

When faced with interesting material that will not reproduce well, I often take the book or journal to class and pass it around. Students are very receptive to this and enjoy looking at the books.

The best illustrations to make overhead transparencies from are woodcuts or etchings. I always try to find illustrations that were produced contemporaneously with the individual concerned; I have a strong personal dislike for illustrations that have been prepared for modern books. However, early etchings can be hard to find. The best advice I have is to be diligent in searching out good sources.

Title Pages

The title pages of important mathematical works often make very attractive illustrations for use in a lecture or class. Often they contain a printer's device that is visually attractive. The titles of the works are themselves interesting. When they are

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in Latin, or French, or whatever language, they are a quiet comment that not everything of value has been written in English or even translated into English. Thus, in some small way, this encourages the study of languages.

On any given title page, there are always several things to comment on, and this should be viewed as an opportunity to present a richer picture of the history of mathematics. Usually the title gives a good indication of the contents, especially the older verbose titles. There may be some words that need explaining; for example, after Copernicus published his *De revolutionibus orbium coelestium* in 1543, the word 'revolutionibus' changed in meaning from revolving (around the Sun) to overthrowing accepted dogma or institutions, as in political revolutions. There may be something on the title page indicating the author's affiliations. This provides an opportunity to explain what such things as 'FRS' mean. In many seventeenth-century books the date is given in Roman numerals, but in an open format, where 'M' is not used but an older variant. This gives the opportunity to discuss the fact that symbolism changes over the centuries. Thus the main point that I wish to make about the use of title pages is that they are not just a way to spice up the presentation with visuals, but that they actually provide an opportunity to discuss things that would otherwise have to be forced into the discussion.

So how does one obtain photocopies of title pages to make overheads from? There is no standard source where all can be found, but there are books that reproduce many of them.[2] These range from David Eugene Smith's *Rara arithmetica* (1908) to *Heralds of science: As represented by two hundred epochal books and pamphlets in the Dibner Library, Smithsonian Institution* (1980). The catalogs of rare book dealers often contain excellent illustrations. Most textbooks about the history of mathematics will contain a few title pages. With the use of a photocopy machine that enlarges these can often be made into excellent transparencies. Research articles on the history of mathematics are an excellent source of title pages and other illustrations. Finally, the ultimate source is rare book rooms. The difficulty with this source is that you will have to go to the place where the book is located and then you will have to gain the confidence of the librarian in order to get a photocopy made. Often, because of the condition of the books, librarians will not allow photocopies to be made. They will probably let you order a photograph of the title page, but this is time consuming and expensive.

Another way to obtain a copy of an text is to use a microfilm of the original. An amazing number of older mathematical works are available on microfilm.

Finally, a last resort for pictures is to use the commercial picture services.

What you need to do is to constantly keep your eyes open for good illustrations. Everytime you look at some historical work, think about the illustrations that are there. Are they suitable for reproduction as an overhead? Will it be useful to you in your class? In a lecture you will give? In some future publication? You should definitely keep notes about where you find good illustrations, for you will not always (ever?) remember where you saw them when you want them.

Oftentimes, one wants a specific illustration and then the problem is more difficult. I have looked hard for a portrait of Robert Hooke, but I am coming to the conclusion that none exists. The illustration of Bernhard Bolzano reproduced in the note about the Intermediate Value Theorem came from a volume of his philosophical writings. So do not restrict your attention to mathematical works when looking for illustrations. When looking for a specific illustration, you should look first in general sources, then in works about the individual, then in their works, and when none of that works, ask a friend or post a note on the internet.[3]

Over the years it is possible to build up a considerable collection of overheads that are useful in a variety of situations. For a given lecture it is

[2] For suggestions on possible overhead transparencies, but primarily restricted to pre-calculus mathematics, see Barnabas Hughes, "Transparencies in the history of mathematics," pp. 187–196 in *History in Mathematics Education. Proceedings of a Workshop held at the University of Toronto, Canada, July–August 1983* [=*Cahiers d'Historire & de Philosophie des Sciences*, no. 21], edited by I. Grattan-Guinness and published in 1987.

[3] There is an email group dealing with the history of mathematics. To subscribe, send email to

`majordomo.maa.org`

consisting precisely of the single line:

`subscribe math-history-list`

Momentarily you will receive a message indicating how to post messages to the list. However, it is advisable to be a silent reader for a while before starting to participate. This will give you the opportunity to judge the level of the discussion.

probably difficult to find, all at once, the 30 or 40 illustrations that you would like to have. This must be done over the long haul.

Straight Text

While I frequently use straight text overheads in presentations at meetings, I almost never use them in class. The reason is simple; when using them you go too fast.

Overheads that involve only text are easiest to prepare, yet are often most poorly prepared. The most important thing to remember is that one has a tendency to reuse the same overheads, so what may have been prepared for a small group, is later used for a large crowd. When this happens, a disaster results.

When using a computer to prepare an overhead, my rule of thumb is that there should not be more than 12 lines to a page. This may see seem like very few lines to you, but remember that often you will want to use the overhead in a large room. If you are going somewhere to make a presentation, you never know what kind of room you will be faced with. Consequently, an overhead is good for an outline of what is planned for the talk, a list of items, a short pithy quotation, and other such items. But resist the urge to crowd in more material. A crowded overhead is a poor overhead.

Now admittedly, I break my own rule. Sometimes I make an overhead of a page of a book, but only when I want to show an effect. I use overheads of whole pages from works of Descartes and Euler to show the notation they used, to show differences between different editions of the books, and to give the feel of the original. But I do not expect the audience to read the material on these overheads. Everything you want the audience to comprehend you must say.

To prepare good overheads takes practice and experimentation. I keep a file of sample computer generated overheads, each marked with the point size and font used. A point size of 14 is the minimum that should be used; 18 or 24 is better. I prefer fonts without serifs. Since all of this depends on which computer system you will use, specific details are probably not too helpful here.

Occasionally, because of time constraints or the use of complicated mathematical symbolism, I draw an overhead by hand. When doing this be sure not to crowd the overhead. Be sure to use a black or blue washable marker. Colors such as yellow, orange, and red look good to you when preparing the overhead, but they cannot be seen by the audience.

Making a Transparency

Let us suppose that you have selected the material that you want to make into an overhead. Probably, you have in mind an illustration that appears in some book. Take it, together with scissors and tape, to the best photocopy machine that is available to you, preferably one that has been serviced recently. It is essential that it be capable of enlarging (and occasionally reducing) the original. Place the book on the photocopy machine. If it a whole page you want, make a copy. If it is just a portion of the page, use the enlarger, for you want the picture on your overhead to fill up most of the page. It may take several tries to get the proper enlargement; another option is to enlarge an enlargement you have already made, but remember that each time an image is photocopied some quality is lost.

Now take the photocopy and trim any dark edges (too often this step is omitted). If the figure copied has a figure number, you probably want to trim it too. That may make the caption that came with the figure off center; so cut off the whole caption and move it over. Then tape the two together at the back. If possible, don't let the tape show. Almost always, the caption will not be large enough to read on the overhead, so enlarge it before using it. Now go back to the photocopy machine and make another copy to see how it looks, for this is the way your overhead will look. Are the various pieces properly aligned?

Often I make a dozen photocopies to produce one overhead. This has a tendency to make my department chair, who is in charge of the budget, quite nervous, but I insist that there is no reason to prepare an overhead if it is not done right. I always win this issue, and so encourage you to persist. This problem can be avoided by preparing overheads in the evening; also your colleagues aren't encouraging you to hurry up. After years of making overheads I have not become more parsimonious with materials. If anything, I am now fussier about how my overheads look and so make more photocopies in preparing them.

When you have everything just the way you want it, then you are ready to make an overhead. You want to use the type of overhead master that can be run through the photocopy machine (the particular kind depends on the model of copying machine you have, but hopefully the departmental secretary will have solved this problem for you).

Under no circumstances should you use the old thermofax type transparency master. They make terrible overheads. When everything is just the way I want it, I make one final photocopy. Then feed in the transparency master and with one more push of the button you have a beautiful overhead transparency.

However, you have not finished. The kind of masters I use have a blank sheet of paper attached to the plastic with some gooey stuff. So I immediately peel off the backing sheet and replace it with the the final photocopy that I made. If you don't do this then you will have to spend time during your lecture peeling the backing sheets from the transparencies (and they make a mess by sticking to other stuff). Other masters have a paper strip down the left side of the plastic. Immediately after making a transparency I cut that off and throw it away; I hate it when people use transparencies where these black strips show on the screen. It is very distracting.

Finally I take that last photocopy and make a note on it where the original came from (if you don't you will be embarrassed when someone asks where you got that picture). Also, I make notes on this photocopy reminding me of things I want to say while the overhead is on the projector. Perhaps those notes are some biographical information about the individual, something on the overhead that you want to be sure to mention, or the translation of some foreign language text. During your presentation, you should place the overhead on the projector and hold the marked photocopy in your hand. If you have focused the projector carefully before your talk, there is no need to look back at the screen. You can face the audience and use the photocopy as a crib sheet to remind you of what you want to say. They will be impressed with your erudition.

Giving Your Talk

Get to the room where your presentation will be made before the audience arrives if at all possible. Check out the projector that has been provided. Put an overhead on the projector, turn it on, focus it carefully, and then move to the back of the room to see how it looks. Have you focused properly? Is it adjusted so that the image fills the screen but does not go off the edge or over the top?

My preference is to have two overheads and two screens. This allows me to leave some overheads up for reference.

If you intend to use the same overhead several times during your talk, it is best to make multiple copies. It is amazing how hard it is to keep track of them while making your presentation.

If you choose to write on your transparencies during your lecture be sure to use washable pens. Don't turn and point to something on the screen; rather place the pen on the overhead to point to what you wish to emphasize. If you point with your hand you will block out too much of the screen.

Don't fastidiously stack up your overheads after you use them. This gives your audience the impression that you care less about them than you do about not having to sort our your overheads later.

After your presentation make notes on how things went. How could your presentation have been improved? How did your transparencies work? Should you have a picture of something else?

Have a great presentation. But remember, good overheads are only a necessary condition. You have to prepare the whole lecture with the same care.

Good luck!

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How To Learn More History

1. READ. The best way to learn about the history of mathematics is to read as much material as you possibly can. If you are active in research, read about the history of your field and also the original papers; you will benefit greatly. All teachers should read something about the history of the mathematics they teach.

It is very rewarding to read some of the professional journals. *Historia Mathematica* has much of interest including information about current literature. *The Archive for History of Exact Sciences* is a very high level journal that contains first class articles. *ISIS* is a general history of science journal with lots of book reviews.

Original sources can be most rewarding. Two of Euler's works are currently available in English from Springer-Verlag: *Elements of Algebra* and *Introduction to Analysis of the Infinite*. I recommend both of them highly.

2. JOIN. To become active in the history of mathematics community you need to get involved with the professional organizations. Here are the two most active ones in the US:

History and Pedagogy of Mathematics (HPM) is an affiliate of the International Commission for Mathematics Education that encourages teachers at all levels to use the history of mathematics to motivate their students. HPM publishes a *Newsletter* that will be sent inexpensively to interested teachers. To receive it send you name and address to Victor J. Katz, Editor, Department of Mathematics, University of the District of Columbia, 4200 Connecticut Ave. N.W., Washington, D.C. 20008. You can also contact him by email:

`VKATZ@UDCVAX.bitnet`

This group meets each year in conjunction with the annual NCTM meeting and every four years with ICME. The talks almost always deal with how to use history in the classroom.

The Canadian Society for History and Philosophy of Mathematics meets annually in late May. The talks at CSHPM are primarily at the research level, but there is much to benefit the teacher. Recently they have been publishing proceedings of their meetings. Joining CSHPM is the cheapest way to get a subscription to *Historia Mathematica*; dues are \$50 per year. You need a recommender to join; you can use my name.

3. ATTEND. Active participation at meetings is the only way to really find out what is going on. You will find the historians to be a most congenial group. Besides the annual meetings of HPM and CSHPM, here are a couple of upcoming special meetings. The first one should be super.

The Eighth International Congress on Mathematical Education (ICME-8), will be held this summer in Spain. It will be preceeded by an HPM meeting in Portugal.

The Midwest Conference on the History of Mathematics is held every two years.

In conjunction with the annual joint AMS/MAA meetings there are frequently special sessions on the history of mathematics. Look for them and attend. You will meet a lot of interesting people this way including many of the leading scholars in the field.

4. WRITE. Don't hesitate to write to historians about their work and the questions that you have. They are happy to send copies of their papers.

I have an email group dealing with the history of mathematics send email to

`majordomo@maa.org`

consisting of the single line

`subscribe math-history-list`

to join.

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Finding Aids

One of the most difficult tasks for anyone interested in the history of mathematics is to know where to look for information on a specific topic of interest. After checking in the general histories of mathematics (Dauben 1985a lists 29 of them on pp. 22–28) and following up the often meager references there, one needs to know where else to look. The items listed below are some of the most helpful sources that I know. I have used all of them repeatedly. It would be impossible to give a comprehensive list. Only things which I have found particularly useful are included.

Dauben, Joseph W.
1985a *The History of Mathematics from Antiquity to the Present. A Selective Bibliography*, New York: Garland, ISBN 0-8240-9284-8.

An annotated classified list of over 2000 books and papers contributed by 49 specialists. The standard for inclusion is high, so this is the first and best place to look for the best literature on a given topic. Expository papers are not usually included, so we need a volume like this for use by students in history of mathematics classes.

Read, Cecil B. and Bidwell, James K.
1976a "Periodical articles dealing with the history of advanced mathematics," *School Science and Mathematics*, **76**, 477–483, 581–589, 687–703. Earlier versions of this, with various titles, can be found in the same journal, 59(1959), 689–717, **66**(1966), 147–149, and **70**(1970), 415–453.

This is a listing of the titles of papers in nine common journals which deal with the history of mathematics at the level of calculus or below. They are grouped under a few large subject headings. This is the best bibliography for students to use when writing papers for history of mathematics classes.

May, Kenneth O. (1915–1977)
1973a *Bibliography and Research Manual of the History of Mathematics*, University of Toronto Press. Reviewed by Grattan-Guinness, *HM* 1(1974), 192–194.

This is an exceptionally valuable bibliography of books and papers, both of an expository and research nature, on the history of mathematics covering the period 1868–1965. It begins with a "research manual" which discusses the basic reference material available (p. 7 ff.), the absolute necessity of some personal system of bibliographic record keeping (p. 12 ff; He recommends slips of paper and discusses how to use them.), and some of the dangers of historical writing (p. 28 ff; For example, the various kinds of plagiarism). It is well worth your time to read these introductions. The bibliography which follows lists about 31,000 entries under some 3700 headings. There are extensive sections dealing with biography, mathematical topics, areas related to mathematics (epimathematical topics), and a classification of papers by historical period. No one interested in the history of mathematics at any level can afford not to be acquainted with this work. The main drawback is that the titles of the papers are not given (to save space) and so you cannot use the inter-library loan service. Unfortunately, Professor May's collection of reference slips was destroyed after he died.

Gillispie, Charles Coulston
1970–1980 *Dictionary of Scientific Biography*, New York: Scribners, 16 vols. Two supplementary volumes appeared recently. See "The *DSB*: A Review Symposium," *Isis* **71**(1980), 633–652. I know of no review specifically of the mathematical portions.

A truly superb source of accurate up-to-date information about deceased scientists. Each article contains biographical information, a sketch of the person's work and a bibliography of primary and (only the very best) secondary sources. This very high level reference book is the first place to look for information about a mathematician. All of the articles about mathematicians seem to be at least competently done; the best are excellent. Volume 16 contains a name and subject index as well as a list of scientists by field. There are also very useful topical essays on India, Mesopotamia, Egypt, Japan, and the Mayas.

Grattan-Guinness, Ivor
1977a "History of mathematics," pp. 60–77 in *Uses of Mathematical Literature*, A. R. Dorling editor, London: Butterworth.

A basic introduction to all aspects of the historical literature. It is a greatly expanded version of this finding aid. Contains, along with much else, a lists of bibliographies, catalogues, and journals for the history of mathematics.

Jayawardene, S.A.
1983a "Mathematical sciences," pp. 259–284 in *Information Sources in the History of Science and Medicine*, edited by Pietro Corsi and Paul Weindling, London: Butterworth.

Another basic guide to searching out information in the history of mathematics. There are quite a few of these, but this one and Grattan-Guinness 1977a are the two best that I know of. These references will help you find more specialized bibliographies and a great wealth of reference works. Other chapters in this volume give valuable information on more general finding aids.

Warning: Whenever you photocopy something you should *immediately* write the full bibliographic details on your copy. Don't trust your memory; at the very best you will still have to retrieve the volume from the library to get the exact title and publication date. I have had a copy of the Jayawardene paper for some time, but until serendipity recently led me to this volume, I did not know where the paper came from. An excellent way of learning about other valuable finding tools is to spend several hours carefully examining the books that are in the reference section of your library.

Many journals have cumulative indices and these can be very useful. *The Mathematics Teacher* has one for volumes 1–58 (1908–1965) and another for 59–68 (1966–1975). Kenneth May prepared the *Index of The American Mathematical Monthly. Volumes 1 through 80 (1894–1973)*. *Isis* has a cumulative bibliography for the first 90 volumes, 1913–1965, and another for 1953–1982; both of these index book reviews, which can be very useful. *Mathematics Magazine* also has an index. You should make sure that your library has purchased these indices. They will be very useful to you and your students.

Regardless of how many bibliographies you have access to, there is still the problem of locating current work of interest. The "Telegraphic Reviews" in the *Monthly*, "Media Highlights" in *The College Mathematics Journal*, "Reviews" in *Mathematics Magazine* as well as "Have You Read" in the *Newsletter of the International Study Group on the Relations Between History and Pedagogy of Mathematics (HPM)*" are all useful ways to keep up with the literature.

Browsing through the current journals is very valuable, but no library has them all, so one must use the abstracting journals. *Historia Mathematica* has both reviews of books and short abstracts of papers. *Isis* publishes a *Critical Bibliography* each year as the fifth number of the journal. It covers the whole of the history of science and also contains an index of book reviews. When reading a book it is always interesting to look up several reviews and see what specialists think of it. *Mathematical Reviews* and the *Zentralblatt für Mathematik und ihre Grenzgebiete* often contain valuable reviews of historical papers. Naturally there is considerable overlap in these four sources, but each contains information missed by the others, so all should be consulted on a regular basis.

Browsing through old journals can also be quite interesting. You will almost always find something of interest. I particularly enjoy old issues of *The American Mathematical Monthly* and *Scripta Mathematica*. When you go to the library to look up an article it is worth while looking through the entire volume it is in; if a journal published one historical paper there are probably others. Never go to the library without taking along a pad of slips (index cards take up too much space) on which to record references. Whenever you see something of interest, either of current interest or possible future interest, you should immediately record the reference on a slip. It is of no help to remember that you saw a fascinating paper if you can't remember where it appeared, so be scrupulous about recording references.

The most recent and potentially most valuable resource tool available to us today is the internet. The indices available on-line through the library vary considerably from school to school. At Bowling Green, I have on-line access to 23 university libraries in the state of Ohio via OhioLINK, to the WorldCat (OCLC-FirstSearch) of nearly 30 million records of books and manuscripts as early as the eleventh-century, to 1.2 million Dissertations Abstracts published since 1861, and to a wonderful periodical list in History of Science and Technology (RGL-Eureka) describing journal articles, conference proceedings, books, book reviews, and dissertations from 1976 to the present. Other indices which I seldom used before such as Art Index (1984-present) and Cumulative Book Index (1982-present) have recently yielded interesting information. Since some of these are commercial adventures, they may or may not be available to you. I encourage you to devote time to mastering the computer library system available to you. This will enable you — and your students — to become a much more efficient library user. If there is a nearby research library you use, ask the librarians how you can telnet to their catalogue so that you can use if from your home base. Much time is saved by searching the online catalogue before you arrive at a library.

The World Wide Web is potentially the most valuable tool available to the historian of mathematics. Rather than try to describe this rapidly changing scene, I encourage you to take a look at my Home Page:

`http://www.bgsu.edu/~vrickey/`

and also to let me know of other internet resources that you find so that I can add them to my Home Page.

Finally, the best way to find out what is worth reading is to talk to people who read a lot of historical work. If your question is not too specific there is probably something in the literature about it. The more specific your question the less likely it is that something has already been written about it. I would encourage you to find someone who shares your historical interests and to correspond with them.

One final piece of advice: get to know your reference librarians. They have a wealth of knowledge and are anxious to share it, especially once they learn you really want to know the answer to your question. The librarians in rare book rooms are often especially knowledgeable. When I go to a rare book room for the first time, I always introduce myself to the librarian on duty, indicate that I am a historian of mathematics, and explain what my special interests are. Time spent talking to rare book librarians may seem extravagant given the shortage of research time, but in my experience it is time well spent. After putting in a call for the book I want to see, I ask where the washroom is, explaining that I want to be sure my hands are clean before I touch any books. I also bring along a handful of sharpened pencils (never use a pen in a rare book room) and a generous supply of blank paper. On a first visit I am very reluctant to ask for photocopies. The purpose of doing these things is that I want to establish that I know the rules of rare book rooms and that I have great respect for the valuable materials that the librarians are willing to share with me.

This file, findingaids.tex, was typeset using TEX on May 30, 1996 at 0:43 A.M.

A Short Bibliography

This bibliography indicates some of my favorite things to read on the history of the calculus. It is specifically designed for people who are just making their way into the history of the calculus.

Boyer, Carl B. Revised by **Uta C. Merzbach.**
1989a *A History of Mathematics*, Wiley. This second edition contains revised bibliographies and revisions of the later chapter. Chapters 17–21 and 25 are the pertinent ones.

Edwards, C. H. Jr.
1979a *The Historical Development of the Calculus*, Springer. A very good sketch.

Euler, Leonard
1988a *Introduction to Analysis of the Infinite*, Springer. Laplace said "Read Euler, read Euler, he is our master above all." Before John D. Blanton translated the two volumes of Euler's 1748 *Introductio* his advice was hard to follow. Now you can easily profit from seeing a master at work.

Gillispie, Charles Coulston, Editor in Chief.
1970-80 *Dictionary of Scientific Biography*, New York: Charles Scribner's Sons. 16 volumes + 2 recent supplementary volumes. The *DSB* is the preëminent source of biographical information, so if you don't know it, go to the library and have a careful look. You will find information on virtually anyone who has made a contribution to the calculus.

Grabiner, Judith V.
1983a "The changing concept of change: The derivative from Fermat to Weierstrass," *Mathematics Magazine*, 56(1983), 195–203. Distinguishes four stages in the chronological development of the derivative: use, discovery, exploration and development, definition. In your reading try to adapt this scheme to the other concepts of the calculus. "Invention of the calculus" is explained on p. 199.

Grattan-Guinness, Ivor (Editor)
1980a *From the Calculus to Set Theory, 1630–1910. An Introductory History*, London: Duckworth. Also available in paper. Six excellent chapters, by some of the best contemporary historians of mathematics.
1987a *History in Mathematics Education. Proceedings of a Workshop Held at the University of Toronto, Canada, July–August 1983.* Published as no. 21 of *Cahiers d'Historie & Philosophie des Sciences*, Paris: Belin. This volume contains several papers that discuss the use of history in the classroom.

Hairer, E. and Wanner, G.
1983a *Analysis by History*, Springer, 1996. This is loaded with information.

Katz, Victor J.
1993a *A History of Mathematics*, Harper-Collins. This and Boyer/Merzbach are the best surveys of the whole history of mathematics available.

Kitcher, Philip
1983a "The development of analysis: A case study," chapter 10, pp. 229–271 in his *The Nature of Mathematical Knowledge*, Oxford University Press. Paperback 1984. A penetrating survey, by a good philosopher of mathematics, treating Lakatos's ideas on Cauchy's famous wrong proof.

Kleiner, Israel 1989a "Evolution of the function concept: A brief survey," *The College Mathematics Journal*, 20, 282–300.

Rickey, V. Frederick
1987a "Isaac Newton: Man, Myth, and Mathematics," *College Mathematics Journal*, 18, 362–389. A survey of Newton's contributions.

Simmons, George F.
1986a *Calculus with Analytic Geometry*, McGraw-Hill. Since one facet of this Conference will be a discussion of how the history of the calculus can be used in the classroom, I suggest you look at this textbook. Most obvious is the long (pp. 763–848) biographical section, but there are numerous historical remarks scattered throughout.

Struik, Dirk J.
1967a *A Concise History of Mathematics*, third revised edition, Dover. If you have time for nothing else, read chapters V–VIII of this. Note how he pays attention to the social history.
1969a *A Source Book in Mathematics, 1200–1800*, Harvard. Reprinted in paperback by Princeton, 1986. The importance of reading original sources cannot be overstressed. The selections and introductions here are excellent. The last half of the book deals with the calculus.

This file, Short-bib.tex, was typeset using TeX on May 30, 1996 at 1:08 A.M.