Writing Faculty Guide

Contains Articles Extracted from

USING WRITING to Teach Mathematics

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for

Staff and Faculty

Department of Mathematical Sciences

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Edited by Andrew Sterrett

THE DEPARTMENT OF MATHEMATICAL SCIENCES

WRITING FACULTY GUIDE

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INTRODUCTION

The USMA Department of Mathematical Sciences has a unique mission. There are few other colleges which require all students to take two full years of mathematics, not including algebra and trigonometry, regardless of their major. Since the United States Military Academy is by tradition primarily an engineering school, what we do supports not just math majors, but large numbers of future mechanical, electrical, systems, and civil engineers. We have the further challenge of teaching those who, by measures of high school preparation, interest, and ability, are decidedly unmathematical. Taking a more far-sighted view, we support the Academy goal of educating leaders of character for the nation. Looking at our mission either way shows a daunting task. To prepare cadets academically for any engineering tracks is hard; to prepare cadets to be leaders of character, with all the attributes that go with that appellation, is very hard. Of course we know the benefits of a sound education in mathematics: students develop the ability to describe the world using mathematical language. This in turn helps them solve complex problems and discover relations that aren't always apparent. We have to teach our students to think logically, use clear definitions, and a generous measure of creativity. These skills are an important part of our mission, whichever view you take.

How then, to teach both those interested and those bored by mathematics, both the able and the barely proficient, and do it well and do it all in the same classroom? One way that is gaining acceptance in colleges and universities nationwide is to ask students to *write* their mathematics. Linking the symbols and notation of mathematics to words (which are more familiar to students) puts mathematics back in the customary world. This helps the mathematically adept student to more fully realize the meaning of and motivation behind the manipulations which he already does well. It also helps the uninterested student see that mathematics is more than meaningless symbols and manipulations.

This faculty guide is an instructor's handbook to explain the why, how, and what of writing in mathematics. The essays included are just a sampling of the literature available in professional journals. Each was selected to provide information and various approaches to using writing in mathematics classes. The essays are by educators from large and small schools, both public and private, across the country. The intent is to persuade you that writing assignments are a worthwhile pedagogical tool. Nothing is meant to be directive in nature; indeed, the nature of writing assignments is such that each instructor can develop his own rules without violating any sanctities.

Section I covers the why of using writing in mathematics classes. The essays discuss mathematics as a language which can be meaningless for some (most?) students and as an incomprehensible process, respectively. Both go on to discuss ways in which writing can show students the meaning, creativity, and power of mathematics. Section II, the how of writing mathematics, gives you some ideas about exactly how, as a math instructor, you can teach and evaluate your students' writing. Section III contains some specific as well as some general writing assignments. Section IV, Student Models, is a selection of actual cadet work. It will be used in the Faculty Development Workshop to teach you how to evaluate writing assignments. Finally, Section V excerpts some writing guidelines used by J. J. Price, Phd., Purdue University, for over ten years. He gives these guidelines to his students at the beginning of the semester so they know the rules before the game is in play. Again, they aren't directive--adapt them to your needs or ignore them entirely.

Section I

Why Use Writing to Help Teach Mathematics?

The two essays in this section argue persuasively that mathematics without meaning--mathematics as empty symbols juggled mechanically--is sterile and as frustrating to teach as it is to learn. Mathematics should be presented for what it is: a language as rich and subtle as the English language it replaces.

Attempting Mathematics in a Meaningless Language

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A colleague recently recounted an undergraduate's complaint about his history-of-mathematics textbook. The otherwise passable text, said the student, was too frequently interrupted by objectionable parades of *symbols* across the page. This student had proceeded with equanimity through pages of English words, which are nothing if not symbols themselves. Thus the problem was not really one of dealing with a symbolic language, but of confronting symbols of an unknown language—empty symbols, symbols with missing referents.

The minimal characteristic of a symbol is that it points beyond itself to something else. A phrase in our natural language has a referent in our own experience: "jet airplane" is a real symbol because we know what a jet airplane is. "Slithy tove" is an empty symbol, having no referent in our experience.

Beginning undergraduates in mathematics need to use the symbol-system of elementary algebra as a language, but it is common for them to experience phrases of the algebraic language as symbols without referents. The student who describes the meaning of the symbol x as "x ... just x" is not apt to find any referent for the symbolic phrase $1/\sqrt{x}$. Even more common is to experience an algebraic symbol as empty in the sense of referring only to other algebraic phrases, so that the referent of $1/\sqrt{x}$ might be simply $\sqrt{x}/x$. If this is a referent, it is of the most trivial and circular kind: according to this understanding algebraic phrases can have no meanings, but only synonyms.

Yet no real language is so impoverished as to describe only its own words. Language has to be *about* something to be real, and even students who are mute in the algebraic language usually believe that it functions as a language. It is to be seen in books and journals, so that evidently it supports at least some kind of communication between at least some sorts of people. Do mathematicians at supper converse in this strange manner? The students' question is not entirely frivolous, for they usually follow it with the *right* question about a language: What do you talk about?

The students whose mathematical activities began these speculations are first-year calculus students at the University of New Hampshire, particularly those who visit a Mathematics Department facility called the Mathematics Center. Some are doing required first-semester remediation in elementary algebra and trigonometry; some are working on projects in numerical integration; some drop in with calculus questions. They afford the Mathematics Center's staff a privileged view of the ways in which large numbers of students address problems in mathematics. Mathematical misconceptions that might otherwise be put down to individual error sometimes command an observer's attention just because so many students display them.

A large part of our students' difficulty with calculus would appear to be language-based. They seem to use their algebraic language almost exclusively in its calculational or syntactic aspect, finding its descriptive, meaningful aspect inaccessible. Thus, for instance, they can complete word problem solutions—once the meaning of the problem has been distilled into an algebraic sentence. They eventually wade through their algebra remediation—provided they concentrate on syntactic examples, those having to do with solving algebraic sentences or rewriting algebraic phrases. They find limits exercises in calculus baffling, since these present only algebraic phrases whose values are to be estimated, whereas the student prefers algebraic sentences which might be solved by grammatical rearrangement. In all these situations the students are willing and able to manipulate the syntax of algebraic sentences, but cannot find or express meaning in the algebraic language.

A frustrating and finally convincing experience in the Mathematics Center began with a certain hint included in a numerical integration problem set. This one-line hint was clearly stated in algebraic terms; yet not only was it incomprehensible to most students, its crucial part was actually invisible to many of them. We will shortly see how this careful attempt to communicate information to students in the algebraic language met with utter failure.

Algebra as a Subset of English

To the extent that students' difficulties are related to their misuse of the algebraic language, it is worth looking carefully not only at student use of the language but at the characteristics of the language itself. It is the assumption throughout this paper that the system of algebraic symbols is not an entirely separate language. Rather, in its descriptive aspect at least, it is a telegraphically-written subset of our natural, spoken language. Sentences written in the algebraic language are to be understood in the natural language: variables, even those denoted by letters, are to be understood as quantity *names*. The calculus instructor who writes "$w = fd$" on a blackboard is simultaneously uttering some English sentence containing the nouns "force," "work," and "distance." Unfortunately, many calculus students who copy "$w = fd$" from the blackboard into their notebooks will thereafter read the sentence literally, with no mental reference to English nouns.

Assuming that the algebraic language is a subset of English, we should note that the inclusion is proper. There is, for instance, no algebraic equivalent to "The dog needs a bath." So a reasonable first question is: which English sentences have algebraic equivalents?

First-language acquisition by young children is a remarkable process, not completely understood and of wide interest among linguists and psychologists. In this connection one linguist writes:

> For each English active sentence with a certain sort of transitive verb (including *eat, bite, catch* ... etc., but excluding *cost, weigh* ... etc.) there exists a corresponding passive sentence with the same verb ... [4]

From our point of view, the striking thing is that the example's excluded verbs are just such as might occur in the word problems of elementary algebra. Hints should be taken wherever they are found, and this one suggests that only certain verbs will occur in English sentences that have algebraic equivalents.

The English sentences that have algebraic translations are those having to do with quantities. They have nouns that represent quantifiable entities, and often their verbs—like *cost* and *weigh* in the example—correspond to measure functions. These words, appearing as verbs in the English sentence, will instead appear as part of a noun phrase in its algebraic version. For instance, corresponding to the verb "weigh" is the measure function $weight_{of}$, so that the English sentence "The dog weighs 75 pounds" has for its algebraic equivalent "$weight_{of}$(dog) is 75."

Producing Algebraic Sentences: Some Ways that Work, and Some that Don't

Inability to extract meaning from an algebraic sentence is one part of our students' difficulty with this essential language. Even more obvious is the converse problem of expressing meaning in the language. This is apparent in their reluctance to attack word problems, where the missing skill is that of summarizing the meaning of the problem in algebraic language. Let us consider first a pair of strategies that do *not* help students produce algebraic descriptions of problems.

The first of these is advice to the student to read the problem carefully. This good advice is redundant, for the cause of failure with a problem is apt not to be careless reading of it. Reading students' incorrect attempts at word problem solutions will often make it plain that they have indeed understood the problem requirements, but have not been able to summarize them in an algebraic sentence.

The other trouble with advice about careful reading is that word problems in elementary algebra and calculus are often chock-full of data. As with a tangled skein of yarn, finding the loose end is more profitable than carefully contemplating the whole tangle. What the student needs is not so much careful attention to the whole of the problem as a way to find the particular piece of data or point of view that will start the problem unraveling.

A second piece of advice commonly given to students is to begin a word problem by giving every quantity in it a letter-variable name. Students seem universally to have accepted this advice; they are everywhere to be seen dutifully initiating problems with glossaries like "let $x = \ldots$, let $y = \ldots$." There are several things wrong with this approach.

What the student needs is not so much careful attention to the whole of the problem as a way to find the particular piece of data or point of view that will start the problem unraveling.

For instance, the names "x" and "y" convey no meanings. Students use literal variables because they think it is the correct thing to do, yet their use starves the student in the midst of semantic plenty. In its descriptive aspect, the algebraic language is part of English, with English nouns available for use as variables. It is pointless to begin a problem with the statement "let x = width of pasture" when the English word "width" would have been a more meaningful variable, requiring no glossary entry at all. The construction of a variable glossary for a word problem is a preliminary task for the student that yields no further insight into the problem. At best it produces a correspondence between English noun phrases and their algebraic equivalents, giving the student something extra to remember.

But the chief reason that glossary-building is nonproductive is that it provides references only to the sentence's noun phrases. With only the nouns in hand, the student will still not know what to say about them. The trouble is that the verb is missing. Until the problem statement has a verb, no collection of noun phrases will lead the student to the desired equation.

Verbs turn phrases into communications. In the construction of an algebraic sentence, the crucial step is to combine noun phrases meaningfully with a verb. The traditional method of constructing an algebraic sentence is to do it wholly within the algebraic language. One names the variables, collects algebraic noun phrases and then (somehow) assembles them with an appropriate combination of the verbs "is" and "exceeds." If this approach were productive, students would be much more successful with word problems than they are.

A better method, we believe, is to assemble the whole sentence in English first. The student then has the advantage of using the familiar natural language for thinking about the problem's summary, and the English summary sentence will emerge already equipped with its verb. Naming the problem variables is deferred until the summary sentence is written.

Verbs turn phrases into communications.

Students need some permission and guidance in beginning problems in this way, but oddly enough they do not need help in supplying suitable variable names. Providing invented names for things is an activity humans, even very young children, are known to do without instruction. [2] In fact, one has only to listen to their everyday conversation to hear students produce variable names: a recent favorite has been "Well ... *whatever*." It does no harm, therefore, to omit the formal construction of a variable glossary, proceed directly to the stage of supplying a sentence, and allow a deferred variable-naming process to occur naturally.

Setting Up Word Problems

To see how problem-solving algorithms based on these considerations might differ from those that are usually offered to students, we outline the Mathematics Center's directions for organizing maximum-minimum problems in calculus.

Students are first asked to be sure that the problem really is a maximum-minimum problem, and to write down in English the name of the quantity to be optimized. This has the effect of encouraging the student to read the problem carefully, yet offers focus to the reading. The student is also told to illustrate the problem with a sketch.

The student is next asked to write a sentence describing the target quantity in terms of other problem quantities. The sentence can be brief, but it should be in English: "*Area* is length times width," or "*Total Area* is area of circle plus area of square," or "*Profit* is income less total cost."

Finally, if the target quantity depends on more than one other quantity, we ask the student to return to the problem for extra information relating these "competing independent" variables. The advantage of describing the target quantity first, and only afterwards considering side conditions relating other quantities, is to keep the problem's main statement central in the imagination. Without some kind of hierarchy among the problem's relationships, the student is apt to be diverted by relationships that are valid but do not lead to a solution.

The statement-first, top-down strategy can quickly produce the problem's central equation, complete with a suitable independent variable. For instance, the following maximum-minimum problem was taken from an old calculus hour examination. The calculus lecturers asked that this sample exam be posted in the Mathematics Center, where it naturally provoked student interest and discussion.

> *Example.* Citrus Hill can expect an average yield of 40 bushels of oranges per tree when it plants 20 trees on an acre of one of its fields. A researcher has learned that each time an additional tree is planted on an acre, the yield per tree decreases by one-half bushel. How many trees should be planted per acre in order to produce a maximum yield?

On a recent afternoon three calculus students and one of the Mathematics Center student workers, a senior mathematics major, spent ten minutes considering how to set up an equation for this problem.

The focus of their discussion was on the quantity to be chosen for the independent variable. Because the problem asked "How many trees . . . ?" the students proposed making $number_{of\ trees}$ the independent variable; but then they were confused by the base condition of the 20 initial trees. The student worker pointed out that 20 trees correspond to 40 bushels per tree, 21 trees to 39.5 bushels, 22 trees to 39 bushels, etc. Considering a small table of this data led the students to an appropriate choice of independent variable.

By contrast, notice how naturally an equation shows itself when we proceed top-down, beginning in English with a description of the target variable $Yield$ which is to be maximized:

$$Yield \text{ is } number_{of\ trees} \text{ times } bushels_{per\ tree}.$$

Because $Yield$ is not yet described in terms of only one independent quantity, we recursively describe the successor expressions $number_{of\ trees}$ and $bushels_{per\ tree}$:

$$number_{of\ trees} \text{ is } 20 + extra_{trees}$$

$$bushels_{per\ tree} \text{ is } 40 - extra_{trees}/2$$

The description of $Yield$ is now

$$Yield = (20 + extra_{trees})(40 - extra_{trees}/2)$$

which the student has probably already abbreviated to

$$y = (20 + x)(40 - x/2).$$

Notice that not only has the problem's central equation emerged naturally, but so has a suitable independent variable.

Maximum-minimum word problems in calculus are more easily set up than are the word problems of elementary algebra. They are more uniform in type, and the quantity to be optimized is usually easy to spot. The word problems found in elementary algebra texts are more varied in kind, so that authors often separate them into types according to the standard formula whose use is anticipated.

To use a statement-first scheme in the solution of a word problem from general elementary algebra, the English sentence should summarize the relationship among quantities after the problem's requirements are met. For instance:

> *Example.* Sonya, Henri, and Felix are grading a stack of calculus quizzes. Sonya could grade the whole stack alone in 2 hours, Henri in 3 hours, and Felix in 4 hours. How long will it take them to do the work together?

Students may remember that this is a standard sort of problem, and that a certain standard trick makes it work out easily. But it can be begun with a sentence summary:

Sonya's portion + Henri's portion + Felix's portion = 1 whole stack of papers.

Each person's portion can be re-described. Sonya's share, for instance, will be 1/2 stack per hour, multiplied by the time the committee spends reading the papers. Rewriting each reader's share in this way leads immediately to an equation in the one variable $time_{spent}$.

Visualizing Problems

Pictorial imagination is a crucial part of mathematical problem-solving activity, and one which many students fail to use effectively. It would be misleading to classify this kind of imagination as "nonverbal" in the sense of being something quite divorced from language. On the contrary, it is worth considering the differing pictorial resources of the two languages calculus students must use, and the situations in which they are apt (or not apt) to picture their problems.

Pictorial imagination is a crucial part of mathematical problem-solving activity, and one which many students fail to use effectively.

The natural language first acquired by young children is rich in the names of familiar objects. Indeed, first books in the nursery are usually simple collections of illustrations of named objects. The profuseness of illustration in small children's storybooks, together with the low proportion of text to illustration, make it clear that the text alone cannot convey the meanings intended. Not only are the pictures an essential part of the stories, but the visual images are actually part of the language itself.

As readers mature, written text supplants much of the illustration in reading material. It is not to be supposed that older readers have ceased to enjoy pictures, but rather that the expanding language acquires more and more of its own visual power. Popular adult paperback mysteries, even those we should describe as graphic, contain no line drawings. They are not needed, because the language itself is visually sufficient. Poetic language is a more extreme example. It can be so visual that added illustration would be not only superfluous but an annoying impediment: "Teintés d'azur, glacés de rose, lamés d'or" is not amenable to illustration.

There is no natural visual world appropriate to the algebraic language. A specifically invented environment, the Cartesian coordinate system, provides the language's visual component. The visual aspects of the algebraic language as they are illustrated in analytic geometry differ in two important ways from the natural world and its richly visual language.

One important difference is in the visual power of the languages. Even with its invented visual facility adjoined, the terse and pithy algebraic language does not enjoy the visual richness of natural language. First-year calculus books are rich in necessary supplemental illustrative figures. As with the pictures in children's books, the figures in a calculus text are not extraneous but are rather part of the language of exposition. The algebraic language itself, at least in its semantic aspects, can require accompanying illustration for completeness.

Another difference is in the syntactic parts of the language that can be pictured. In natural language, nouns and noun phrases are illustrated: "the boy in blue," "the mountains at sunrise," "two spheres." On the other hand, in the coordinate geometry of, say, two dimensions the primitive recorded objects are ordered pairs of numbers. As the recorded pairs have coordinates with some relation to each other, it is essentially algebraic sentences, not algebraic noun phrases, that are displayed. The sentence $y = x^2 + 3$ can be illustrated, but not the phrase $x^2 + 3$.

Regions in the coordinate plane are visible only in the sense that we display the sentences that form their boundaries. With a distance measure applied to points on the bounding graphs, we can ascribe geometric notions like height and breadth to their enclosed planar regions. But the regions themselves are still constructs of the imagination, and it can be nonintuitive to students that regions in this entirely artificial coordinate plane are described and measured as if they had natural-world reality. A certain level of mathematical indoctrination is required before the Cartesian coordinate system achieves the reality of a landscape in the natural world.

First-year students often have trouble visualizing geometric applications of integration such as areas, volumes of revolution, and centroids. With surprising frequency students attempt these problems without any sketches at all, simply inserting likely-looking algebraic phrases into standard formulas that they have memorized. Thus they delete from their algebraic language its necessary added visual component, with results that are usually disastrous.

It should not be imagined that students use memorized formulas in this unilluminated way because they cannot picture the requirements of problems. Many of them are studying physics and engineering, where presumably they are used to problems that must be visualized. Indeed we have watched them do their trigonometry remediation and seen their many sketches showing flagpoles atop buildings, forest rangers viewing fires, and brave gunners aiming cannons at the castle of the evil Count. Even when they can go no further with a word problem, students are apt to illustrate it.

There seems to be a difference between those problems for which our students consent to make sketches, and those which they begin without any illustration. Students do generally make sketches for problems that originate in their natural language. They sketch views of their natural world, or of some imagined variant of it. They correctly see illustration as one of the stages in converting a natural-language problem for algebraic resolution.

It is chiefly when the problem both originates and is resolved within an algebraic framework that students slight the task of sketching. This should not be surprising. The sketch must be done within the algebraic language's less familiar visual environment, and drawing it is a curve-sketching task in analytic geometry. The techniques for this are probably not yet part of the student's easy competence.

Geometric applications such as volumes-of-revolution problems usually originate in the context of analytic geometry and in the language of algebra. Not only are these problems subject to the difficulties just noted, but they reach the student with associated standard formulas already provided. With both the problem and its resolving formula written in the algebraic language, the student finds it reasonable to proceed by transcribing algebraic fragments from the problem into the formula.

Encouraging Visualization

Considering the visual properties of their languages suggests some general ways of fostering mathematical visualization by our students. We need to encourage students, both directly and indirectly, to make and interpret sketches in order to do their homework problems. Sometimes the most direct route is appropriate: students need to be directly told to do this. But we also need other ways to interrupt the short circuit from algebraically described problems to algebraic language formulas, in which the student merely transfers phrases from problem to formula.

One simple approach is to use fragments of the more visual natural language in both summary formulas and their accompanying sketches. It may be less elegant to describe an element of volume as $2\pi \cdot radius \cdot height \cdot thickness$ than to describe it as $2\pi p(x)h(x)dx$ or $2\pi p(y)h(y)dy$, but it's a lot clearer. The natural language description leaves it to the student to decide whether the volume element's "thickness" corresponds to a differential in x or y, and then to describe "radius" and "height" in terms of that same variable. This procedure, admittedly wordy, nevertheless shows the student the decisions that actually have to be made in setting up the integral. By contrast the fragment $p(x)$ in the standard algebraic language neatly but tacitly conveys that since the variable of integration is x, the radius of the shell will have to be described in terms of that variable alone. The message may simply be too tacit for the student reader, who often assumes that $p(x)$ represents some algebraic expression already specified in the problem. Good students have told us that they were confused about what to "put down for $p(x)$."

But we also need other ways to interrupt the short circuit from algebraically described problems to algebraic language formulas, in which the student merely transfers phrases from problem to formula.

Students find it important to memorize formulas. It is an activity we can count on them to continue. It will thus be advantageous to encapsulate more meaning into the formulas they are going to memorize anyway, and we have just suggested that one way to do this is to use natural language fragments in their standard formulas. Another way would be to place a relevant sketch with each formula. This suggestion may sound supremely redundant, considering the multitude of sketches already in calculus texts. However, it has become standard practice for texts to appear with important facts and formulas highlighted on the page in boxes of some second color. These are the things the students memorize, and if a formula is in a colored box then a sketch of its application ought to be there also.

Both these suggested remedies have the flavor of a quick fix or patch; and to be sure, there are times when a well-chosen patch is needed and appropriate. But another view of our possible responses begins when we compare the sketches the students make with those they omit. They will avoid the curve-sketching involved in illustrating a volumes of revolution problem, where they would not hesitate to sketch the natural-world dimensions involved in a maximum-minimum problem.

The reluctance of students to make sketches within the coordinate plane's geometry probably has no quick-fix solution. To the extent that students have trouble because they cannot visualize geometric constructs in the coordinate plane, the remedy is more time and effort invested in the teaching of analytic geometry.

Names and their Symbolic Referents

We have considered several aspects of the algebraic language that calculus students find necessary to use but difficult to use meaningfully. The central thread of this troubled web is their attempt to use algebraic names divorced from their symbolic referents.

Earlier we said that the minimal property of a symbol was its ability to point beyond itself to something else. A stronger property would be for the symbol to enjoy some participation in the thing to which it points. A national flag participates symbolically in the dignity of its nation, making it a symbol in the strong sense. By contrast, the uniform yellow color of school buses is a weak symbol, simply a convenient way of making the vehicle obvious and easy to recognize. People are offended to see the flag used in inappropriate ways, whereas they would be merely surprised to see a car painted school-bus yellow. The stronger the symbol, the more troublesome is its use apart from its proper referent.

By the time the encoding of an English sentence into the algebraic language is complete, all its nouns will have been treated in this way. Each will have had applied to it some measure function whose value is a real number. The English sentence's verb may appear as a measuring modifier in an algebraic translation.

Since the noun phrases as they appear in the algebraic version are values of measure functions, the verbs that can occur in the algebraic translation correspond to relational operators. Basically only two verbs will appear in algebraic sentences: they are *is* and *exceeds*. Combinations and rearrangements of these, such as "less than or equal to," are grammatical variations of these two verbs.

Once the English sentence has been encoded algebraically, it has become an algebraic statement relating real numbers. It may have become, for example, the desired equation that summarizes some given problem. Rewriting the algebraic sentence according to the syntax of the algebraic language is a process that amounts to solving the equation, for the grammatical transformations appropriate to the algebraic language are simply calls on the axiom system of the real numbers.

Some Semantic and Syntactic Peculiarities of the Algebraic Language

Every language has a *semantic* component—its descriptive aspect, the part of the language that carries meanings—and a *syntactic*, or grammatical, component. In the case of the algebraic language there is an asymmetry between the semantic and syntactic components that is clear, but worth explicit mention. The language of elementary algebra, being a shortened version of our natural language, borrows the natural language's semantics. The semantics of the algebraic language is not fixed: w might mean "work done" or "weight" or something else entirely. Nevertheless, whatever specific meaning we assign to w will be a meaning from our natural language.

By contrast, the syntax of the algebraic language is that of the real numbers. Sentences in the algebraic language are subject to grammatical transformations which reflect the arithmetic properties of the real numbers. Indeed, the algebraic language is invoked in problem solving precisely in anticipation of grammatical transformations on an algebraic phrase or sentence.

The syntactic power of the algebraic language, providing as it does the means of solving problems, is the source of its importance. Unlike the grammar of a child's first language, the grammar of the algebraic language is explicitly *taught* to students. Most of a course in elementary algebra represents instruction in its grammar. It is hardly surprising that students use the language of algebraic symbols in a primarily grammatical, algorithmic spirit.

The algebraic language's peculiar divergence of semantics and syntax, with its semantics reflecting the natural language and its syntax deriving from the real numbers, sets up an interesting trap for the student who must find meaning in the language. To see where this bind is, we must consider what a meaning might be, and in particular what it might be in the algebraic language.

Meanings of Algebraic Sentences

There is great difficulty in specifying just what meanings are, and we will have to be satisfied with approximate and philosophically fuzzy descriptions. For instance, even if we could confidently specify the meaning of a simple declarative sentence like "The grass is green," we might still be hard put to describe the meanings of related queries ("Is the grass green yet?"), commands ("Green up, grass!") or phrases ("the green grass").The meaning of a simple declarative sentence—or, at least, something very close to its meaning—is often taken to be its *intension*. An intension is a function mapping "possible worlds" onto the set $\{T, F\}$ of truth values of sentences. Each of us at each moment of our lives inhabits a different particular world, so that by a possible world we should understand the totality of our views, knowledge, and experience at some instant. The intension of the sentence "The grass is green" is a function which, applied to a person's own world, produces one of the values $\{T, F\}$. It is the means by which we verify that sentence in our own experience.

A different example may make this notion more appealing. Suppose we are computing values of the expression $3x + 4y$, and we have integer values of x and y stored in a symbol table. How shall we describe the meaning of the expression $3x + 4y$? It should be distinct from any one of its values, which are integers, for it depends on the stored values of x and y. It makes sense to take the expression's meaning to be a function, mapping possible symbol tables into the set of the expression's possible values. Now if we take our own knowledge and experience to be a kind of grand "symbol table" and the values of sentences to be their truth values, we arrive at something very like the idea of the intension mapping.

If this definition of meaning seems scholastic and ethereal, imagine our surprise when we saw calculus students in the Mathematics Center acting it out. What follows is a more complete description of the failed hint, earlier mentioned, that was included in a problem set on numerical integration. The thing to notice is that when students were presented with an algebraic statement meant only to convey information to them, their response was to attempt verification of the sentence. Since the means of verification in the algebraic language are ready to hand, and are syntactic in nature, the very attempt to access the statement's meaning detoured the students into its syntax.

The hint was intended to impress upon the students that error estimation in Simpson's rule requires use of the integrand's fourth derivative, whose form may be extremely awkward. At the same time the students were offered an upper bound for the unwieldy derivative:

Hint:

$$
\begin{aligned}
&|f^{(4)}(x)| \\
&= \frac{|(1+\cos x)(1-5\cos x - 2\cos^2 x) - 3\sin x(5-\cos x)|}{(1+\cos x)^2}. \\
&\leq 34
\end{aligned}
$$

We soon knew that we had done the wrong thing. Repeatedly students told us they couldn't understand the hint. Many students immediately set about calculating an upper bound for the ungainly expression in the hint. Very few of them took any note of the number 34 offered, and a surprising number told us they had not even been able to see it. Confronted with information written in the algebraic language, the students instead tried to compute something with it: in attempting to understand the hint they were diverted from the language's descriptive aspect into its algorithmic one.

Eventually, of course, the students short-circuited their problem. The student grapevine is very efficient, and the word went out, "Don't even think about it, M is 34." A subsequent edition of this problem set contains the same hint written partly in English, which has settled the difficulty.

Names in general have historically been seen as important symbols, sometimes having been extremely strongly linked to the persons or things named. The use of a name without respect for its symbolic linkage has been found offensive in different ways in various circumstances and cultures.

It is worth remembering that mathematical naming is part of a wider and older practice, long associated with peculiar symbolic powers and hazards. An algebraic name, once attached to a number, participates in the algebraic syntax exactly as its referent number would: the strength of its symbolic connection with the named quantity is what makes the name so useful. By the same token, that very symbolic strength makes an empty use of an algebraic name worse than useless. For our students the vacuous use of algebraic names is ruinous, reducing their understanding of mathematics to meaningless and formal exercises on sets of letters.

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Writing in Mathematics: A Vehicle for Development and Empowerment

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For me mathematics is most like:

> an assembly line where a large group of people perform the exactly same task day after day, year after year.
>
> a giant classroom with millions of men reciting the Pythagorean theorem.
>
> the military draft: the amount of interest I have for math is comparable to how much soldiers like the thought of dying.
>
> a closed door: all the information is there, only I don't have the key.
>
> the Sahara Desert: I wander about aimlessly, trying to find the right direction, yet always being fooled by mirages.
>
> quicksand: I find myself drowning in a mass of equations and variables, finding that the more I struggle, the more I drown.

These are some of the metaphors students chose for mathematics early in my Writing Seminar in Mathematics course. Their metaphors make it clear that the students who take the course enter reluctant to approach mathematics, seeing it as mechanical, rote, intimidating, and feeling powerless in its presence.

I offer the course at Ithaca College, a private, co-educational, four-year comprehensive college in upstate New York. The students are entering freshmen who are better-than-average writers, and who want to avoid a more traditional mathematics course. None of the students are mathematics or science majors. Most have interests in the social sciences or the humanities, but have not yet declared a major. The students vary in their mathematical background and confidence, but few believe that they have any mathematical skill or competence.

The goal ... is to change these students' conceptions of mathematics as a discipline and of themselves as learners of mathematics.

The goal of this paper is not to describe the Seminar so that the reader can reproduce it, but rather to entice you to try similar writing assignments in your own courses. I will provide five mathematical situations, each of which could be used in many different mathematical settings, and discuss how these examples, used in the context of cognitive-developmental theory, can become catalysts for changing students' conceptions of mathematical knowledge. Examples of students' writing, which illustrate their mathematical empowerment, will be shared as well.

The goal of the Writing Seminar in Mathematics is to change these students' conceptions of mathematics as a discipline and of themselves as learners of mathematics. I accomplish this by having them participate in mathematical situations, discuss their ideas both in small groups and with the whole class, write informally about their ideas and the process they used to generate those ideas in a journal, and then to express their ideas more formally in well-developed essays.

Everybody Counts: A Report to the Nation on the Future of Mathematics Education, released in January 1989, presents a plan for radical change in mathematics education from kindergarten to graduate school. The report stresses that "students learn mathematics well only when they *construct* their own mathematical understanding" (p. 58). In the Writing Seminar I try to create situations in which students can start with their own ideas and use these ideas to construct their own understanding. Their informal writing in their journals allows me to participate in this process.

The report compares mathematics to writing, stating that:

> In each, the final product must express good ideas clearly and correctly, but the ideas must be present before the expression can take form. Good ideas poorly expressed can be revised to improve their form; empty ideas well expressed are not worth revising. (p. 44)

Many mathematics students believe that they are wrong unless their work is perfect. They do not see mathematics as a place for revision and refinement. They are, therefore, reluctant to take the first steps to mathematical thinking.

I believe that mathematical thinking is stimulated by instances of surprise, of contradiction, of believing, and of doubting. Therefore, I provide experiences that focus on a series of mathematical events particularly designed to raise questions and generate mathematical thinking. Students are encouraged to explore the situations, generally in small groups. They realize quickly that I am a source of ideas, hints, and encouragement; but that I will not give them answers. I help these students become aware of their own experiences with mathematics, accept that they can and do have ideas and intuitions about mathematics, and validate these ideas and intuitions through the use of mathematical theory.

Many mathematics students believe that they are wrong unless their work is perfect. They do not see mathematics as a place for revision and refinement.

Students write about their experience with these events in a journal which I read and comment on every other week. My comments support the students' ideas and their struggle to make sense of the material, raise questions of clarification, try to lead students to see their misconceptions and consider alternative points of view, and invite them to think more deeply about mathematics. While I suggest journal topics to students, I also encourage them to use the journal in their own ways. Each student writes two two-page entries per week. I try to develop dialogues with the students urging them to respond to my questions and comments regularly. (For an account of one student's experience in using a journal in this course, see [4].)

The Developmental Process of Change

Students seem to go through a developmental process in changing their view of mathematical knowledge. What I am seeing as I read the words my Writing Seminar students write indicates that the first step in a changed belief about mathematics comes when a student begins to own his or her own ideas. This seems to involve acknowledging that mathematics was and is made by people. This happens before students become comfortable with the theory, the algorithms, or the procedures that are so necessary in mathematics.

This process is supported by cognitive-developmental theory. The students enter my course believing that mathematics is something that they can have no ideas about; that every question has an answer; that every problem has a solution. Teachers know and teach these truths. Students learn mechanically, following the rules precisely. The excerpts from my students' metaphors for mathematics listed at the beginning of this essay reflect this view, which is termed "dualism" by William G. Perry, Jr. [8, 9] and "received knowledge" by Belenky, Clinchy, Goldberger, and Tarule [2].

I create an environment in this course that helps the students become aware of both diversity and uncertainty in mathematics and gain a more personal, subjective view of mathematics. At this point, their own experiences become worth considering. They still believe, however, that knowledge is the product rather than the process of the inquiry; but they can now learn from the experience of peers, for peers as well as authorities (teachers, textbooks) have ideas worth hearing; also they begin to mold ideas, but often without being willing or able to defend them. This new phase is called multiplicity, or sometimes personalism, by Perry, and subjectivism by Belenky et al.

These students next become aware that intuitions may deceive; that some truths are truer than others; that truths can be shared and expertise respected. They become aware that it is the process of constructing knowledge that is important. They move away from subjectivism, learning to respect established procedures and their own insights, listening carefully to other points of view while evaluating their own. The course helps students move through this process in their view of mathematics. Through this process students become empowered and then become less fearful of the procedures and techniques needed to do mathematics.

Reclaiming Intuition

The following three mathematical situations are used early in the course to help students become more subjective about mathematics, and to acknowledge that they can and do have ideas about mathematical questions and that many of these ideas are valid. Students seem to reject their own thinking when they believe that mathematics must by learned in a rote way. These situations are designed to help students to reclaim their intuition and common sense about mathematics and to enjoy the sense of empowerment and confidence that they feel when this happens.

Situation One

The first problem I ask the students to think about is:

> *The Handshake Question*: If six of us wanted to meet by shaking each others' hands, how would you envision the number of handshakes?

Please, before you read on, stop and think about at least two ways that you would respond to this question.

I plan this as an early experience for several reasons: with a little time and encouragement everyone has an approach; a formula does not usually come to mind; it doesn't look like a traditional school problem and so seems more approachable; and the meaning of "handshake" is open to interpretation.

Students think about the question on paper individually. I encourage thinking in their own style, insist that we all remain quiet so everyone can get an idea on paper, and urge those with one method to find a second or third. The discussion focuses on the variety of methods people have chosen. (See [3, pp. 20–21], for a few of the possible methods.) We get as many of the methods on the board as possible and as the list grows someone will ask which one is "right." This leads to a discussion of the virtues of different methods, which will be interrupted by someone wanting to know which answer is correct.

By this time we will have at least the following answers: 6×6, 6×5, $(6\times5)/2$, and $5+4+3+2+1$. We discuss what each is counting. For example, in 6×5 each person initiates five handshakes, one with each other person. A pair of people will shake hands twice, since each of the pair must initiate a handshake. A discussion of the importance of definitions ensues, since some want to count handshakes this way and others do not. I do not encourage consensus, but rather have each student write the following journal entry outside of class:

> *Journal Entry*: Give two methods you used to find the number of handshakes for 6 people. Give two additional methods discussed in class. Which of these methods do you like best for finding the number of handshakes for 6 people? Why? If you had to find the number of handshakes for 17 people, which method(s) would you use? Why? Build a table with a column for the number of people and another for the number of handshakes needed for them to meet each other. Use 1, 2, 3, 4, 5, 6, 7 in the number of people column. Look at the handshakes column. What are the next three numbers? Why? If you needed to find the number of handshakes for 100 people, how would you do it? Is there anything else you want to say?

This handshake activity could be used in any class where you want to discuss mathematics as a search for patterns, where you want to stress the need for definitions, where you want to encourage multiple methods, or where you need to generate the formula for the sum of the first n positive integers.

Students find it empowering to see so many different methods; for many the empowerment is to see their own method among them. Many students have told me that their mathematics classroom survival strategy has always been to wait for the answer to be presented. These students need to be encouraged to try to solve this problem for themselves; they will try enthusiastically when given "permission."

Situation Two

To help students think more deeply about what they are doing when solving a mathematics problem or using an algorithm, I often ask them in writing to explain a concept to a close friend not in their class, to a parent, to a roommate, to a younger sibling, or to a sixth grader. I want them to use their own words, and to avoid mathematical jargon that is not their own. Try having your students explain a concept to a peer. You will learn what they really understand about the concept. This is an appropriate activity for any mathematics class. In the Writing Seminar, the assignment is:

> *Fraction Paper*: Write a paper explaining to a sixth grader both how and why we add (or multiply) numeric fractions the way we do.

Students remember quickly how to add (or multiply) fractions, but are convinced that they were never told why and cannot figure it out. By working cooperatively in small groups, asking questions, and writing drafts in their journals, they develop good explanations.

hey are again empowered through developing their own mathematical ideas and explanations, by understanding why we multiply ractions as we do, and by realizing that they have not all chosen he same explanations to use.

SITUATION THREE

emphasize examples of the Fibonacci sequence and the golden atio. Students are surprised that these numbers appear in situaions that seem to be totally unrelated. They are also surprised, ifter counting spirals on pine cones, pineapples, and artichokes, hat these numbers occur in nature. One of several problems that use is Fibonacci's rabbit problem:

> *Rabbit Problem*: A pair of rabbits one month old are too young to produce more rabbits, but suppose that in their second month and every month thereafter they produce a new pair. If each new pair of rabbits does the same, and none of the rabbits die, how many rabbits will there be at the beginning of each month? Work the problem until you can predict the number of rabbits at the beginning of the twelfth month.

Students' Changing Conceptions of Mathematics

As I have indicated, my goal in this course is to change my students' conceptions of mathematics as a discipline and of themselves as earners of mathematics. Part of this change is to reclaim their ntuition and common sense about mathematical ideas, an intuition which is lost when they come to believe that their own thinking is nappropriate in a mathematics classroom.

After a number of different mathematical experiences including the handshake question, the fractions paper, discovering the Fibonacci sequence and golden ratio in several settings, and reading *Flatland* 1] have been completed, I give the following writing assignment.

> *Change Paper*: Write a paper describing a specific moment when you realized a change in your thinking, in your mode of acting, or in your feeling about a mathematical question during a classroom mathematical episode.

Students are urged to develop their ideas informally in their journals before preparing a more polished paper.

One student, whom I will call Janet, chose her experience working on the rabbit problem as her moment of change. She first talks (in her journal) about the way she approached mathematical problems before the episode with the rabbit problem:

> Immediately before the rabbit episode, I tensed at the thought of doing a word problem. It's a mental block, almost. I see a word problem, and I automatically prepare myself for confusion. Usually when doing problems in class, I can follow the solution—as long as someone else does it first. (This applies to all problems, not just word problems.) So when approaching the rabbit problem, I figured that I'd attempt to do the problem, not be able to, and then wait for the answer to be explained to me so I'd have some idea of what was going on. I realize now this was a negative attitude to take, but then it didn't seem like an "attitude" at all—it was simply the way I had always approached mathematics.

Next Janet describes (in her journal) her experience working on the problem:

> What is the problem asking? What necessary information is given? At first the problem didn't seem too hard—each month a new pair of rabbits is produced from the existing pairs. The number doubles every month. But as we started discussing the problem out loud in class, I realized that I was not allowing for the one-month-old non-reproductive stage. I tried working with pictures again, as was being demonstrated on the board, but it seemed to be getting too confusing for me to follow.
>
> I recalled that a chart had helped in the tiles problem, so I decided to make a chart. Making the chart was easy to do—the number of newborns the month before was the number of one-month-olds for this month; and the number of adults and newborns was always the same. As I worked on the chart, it was like a light shining through the leaves of a dense forest. (I know it's a lousy simile, but that's really how it was.) I saw. I knew. I wanted to raise my hand and shout out, "I have the answer! It's Fibonacci!" It seems really silly now, but at the time I was ecstatic.

Finally, Janet describes (in her change paper) her thoughts and feelings after the episode:

> I felt proud that I solved the problem alone, but something else increased my confidence. My friend, Ginger, who I consider quite intelligent, showed amazement when I solved the problem so accurately. She still didn't understand how I arrived at the solution, and she asked me to explain. Shocked, I told her that I probably needed more math help than she did. However, she insisted, so I attempted an explanation. As I explained the solving process, I found that I truly understood the problem. Helping Ginger understand it made me feel confident, not just about the rabbit problem, but about the whole mathematics subject. I never helped other people with math before; they always helped me. This new role gave me confidence that never before existed.
>
> After this event, math lost the mysteriousness and impossibility I thought it previously possessed. I lost some intimidation and frustration that I felt when dealing with math.

Janet expected not to be able to do the problem, but she did try a procedure, using a chart, that had worked for her small group on an earlier problem. Not only did she come up with a solution, but it was more satisfying and more efficient than the one being developed on the blackboard. Her sense of feeling empowered by her success overwhelmed her—first when she recognized she had a solution, and second when she realized she could explain it to someone else.

Students seem to reject their own thinking when they believe that mathematics must by learned in a rote way.

A second student, whom I will call Lee, wrote his change paper about his experience writing to a sixth grader about how and why we multiply fractions the way we do. First Lee describes (in his journal) his view of doing mathematics before he worked on his fraction paper assignment:

> Math is done in a strict, step-by-step process in which one missing, illogical, or wrongly done step will ruin the entire problem. Finding the correct answer in math is the main thing, while using the taught method to find that answer is also important. Doing mathematics can result in a precise answer or an estimate but it is not really a thinking process. Rather it is a process of identifying, comparing, and doing a problem in relationship to that identification and comparison. If you got the problem wrong there are no "ifs," "ands," or "buts"; you got it wrong. If you got the problem right and used the correct

method, you got the problem right. The math process is one in which all attention is focused on a narrow subject. My mind is not allowed to create, wander, or think about doing the problem. My mind says, "Compare this problem to others that are like it and base your answer on the way you found your answer to that other problem."

Next Lee describes (in his change paper) his experience working on the assignment:

> We worked in small groups first and I extended this group activity to my individual work. Surprisingly, the dreaded "identifying and comparing process" did not appear in my figuring of the problem. I had no example to follow. I could not merely solve the problem using a previously specified method.
>
> I like writing mainly because I receive satisfaction when I finish a solidly written piece. While writing I am, at the same time, thinking and understanding what I'm writing about. Until the fractions paper I could not say the same about doing a math problem. I thought out and understood what I was doing while I was figuring how and why we multiply fractions the way we do. I felt like I was writing my own chapter of a math textbook.

Finally Lee describes his feelings and thoughts after the episode with the fractions paper. The first quote is from his change paper and the second is from his journal.

> My doing the fractions problem well produced a satisfaction that sparked my interest in math. Furthermore, because I did the problem without referring to any examples of any supposedly proper methodology, I controlled the problem. I chose to take the problem in the direction I did. Finally, the ability to be creative made me want to do the problem and do it well.
>
> Math can be more than just identification and comparison. Math can involve letting the mind wander, create, and yes—think. My mind can say, "Well why don't I try this another way so I can draw up new conclusions or relationships." Math can include more than just numbers and formulas.

I like writing mainly because I receive satisfaction when I finish a solidly written piece. While writing I am, at the same time, thinking and understanding what I'm writing about. Until the fractions paper I could not say the same about doing a math problem.

For Lee mathematics had been a subject in which he did not think, but rather identified and compared. As he worked on the fractions paper, he realized that he would not be given a model; that, indeed, he was being asked to think the problem out for himself. To think, to develop his own ideas, to come to understand the process of multiplying fractions in his own way were empowering and satisfying experiences for Lee.

Beyond Intuition

The early situations in the class are designed to get students to accept and reclaim their intuition and common sense about mathematics, an important step in changing their views of mathematics. However, in mathematics one's intuition must be supported by theory, logic, and data. The theory, logic, data, and the intuition must be evaluated.

After students have gained some confidence and courage to try mathematical questions and to share their own mathematical ideas, I use the following question, The Belted Earth, to generate a conflict between their intuition and the mathematical theory developed to solve the problem. This question helps make students aware of this conflict and urges them to resolve it for themselves.

Situation Four

Please, before you read further, read the Belted Earth question and resolve it to your own satisfaction.

> *The Belted Earth*: In the following question focus first on your initial intuitive response, only then think about a solution.
>
> Think of the equator. Put a flexible steel belt around the earth at the equator so that it follows exactly the contours of the earth. Now add forty feet to the length of that belt and arrange it so that the belt is above the equator for its entire length. The belt still follows the contours of the earth at the equator and is raised above the equator by the same distance at every point.
>
> The question is, what will fit between the earth and the belt? That is, what is the distance between the earth and the belt?

Most students (in fact, most people) believe only something very tiny will fit under the belt and are surprised by an answer of more than six feet. Following an extensive discussion of both intuitive and algebraic solutions, students are asked to write the following journal entry.

> *Journal Entry*: Describe your intuition about the Belted Earth question. Discuss the theory developed in class. Which one do you believe? How are you resolving the disparity between your intuition and the solution developed in class?

I read and comment on this entry extensively. Students now have an investment in understanding mathematical theory and want to make sense of this surprising result. They want to understand what pi is, why we don't need the actual circumference of the earth, and why adding 40 feet to any circumference yields the same result of more than six feet. See [3, pp. 22–23] for a discussion of the development of this problem and the responses to it by a group of intellectually able, adult women.

Situation Five

A second situation designed to help students acknowledge their intuition and then look to theory, logic, and data to evaluate it, is the extensive mathematics unit developing the notion of "straight on a sphere." Students begin with their intuition about "straight" on a curved surface, but then are encouraged through the "believing game" [6, pp. 147–191] to consider the perspective of someone else who might have a different belief than they do. They participate in the development of some mathematical theory and at least one mathematical proof, and develop in their journal writing their understanding of these mathematical ideas. The following describes the sequence of activities that form this unit.

Straight on a Ball

First, in one class we discuss the many uses of the word straight in our general language and in our mathematical language. I then ask the students to consider whether or not we could have a straight line on the surface of the large rubber ball I hold before them. We have a lively discussion with many ideas both pro and con. Then I ask them to write a journal entry explaining what straight means to them and discussing whether or not we could have straight lines on the surface of the ball (or the earth).

Second, I read and comment on their journal entries about straight on the earth. My comments primarily ask for clarification, especially when some of their statements are contradictory. Students tend to discuss one or more of the following:

> Straight applies to lines on the plane and not to lines on curved surfaces.
>
> Straight is the shortest distance between two points and so can exist on a sphere.
>
> Straight is a matter of perspective. A student will talk about a straight line on a sheet of paper, and ask if it loses its straightness when the paper is bent. Another student will argue that if you look directly at the equator drawn on a globe, it will look straight, but if you turn the globe slightly, it will appear curved.

Third, I ask them to reread their own journal entry and then in small groups to play the believing game [6, pp. 147–191], believing that there are straight lines on the surface of the ball. While many do not hold this belief (a straight line is defined on the plane!), they work hard considering what a person who does believe it would have to say. (For example, the person might say that there must be a shortest distance between points on the sphere.) Following the small group and whole class discussion of these beliefs, we agree to define great circles as "straight" lines on the surface of a sphere.

Fourth, we discuss the development of spherical geometry as mathematicians have described it, going back to Euclid's postulates. We see triangles with an angle sum greater than 180°. You can watch the ripple of surprise/disbelief move through the class when you produce (using rubber bands on a ball) a triangle with three right angles clearly defined. We then consider properties we retain and properties we lose as we move from geometry on the plane to geometry on the sphere. We develop the formula for the area of a spherical triangle using Jeffrey Weeks' method [10, pp. 137–148].

Fifth, in their journals, I encourage the students to consider what might be "straight" on other surfaces, like a cone, a cylinder, or the walls of our classroom.

Sixth, in their journals, I ask them to consider why they learn only Euclidean geometry since we do, in fact, live on a surface closer to a sphere than a plane. I ask them to make that consideration in light of what we have discussed as a class and of their own experience.

Seventh, I ask them to consider the historical development of spherical geometry in response to the question, "Was spherical geometry invented or discovered?"

Lee was reluctant to accept the idea that "straight" could apply to lines drawn on curved surfaces. We developed a dialogue in his journal concerning "straight on a ball" that began when I had trouble understanding where he stood on the question and needed clarification.

DOROTHY. Can you have straight lines on the surface of the ball?

LEE. No. It is impossible to have straight lines on the surface of a ball because even the slightest of curves cannot accommodate a straight line. There is no part of the surface of a ball that is not curved. There is the possibility that a small marking on a ball may appear like a straight line. But this is only appearance. Sometimes the human eye alone cannot perceive accurately between straightness and roundness.

DOROTHY. Invent a new word for the thing you would consider the closest that you can come to "straight" on the surface of a ball? What properties would these things need to have?

LEE. The closest thing to "straight" on the surface of a ball is called "stirct." If something is stirct it must follow the shape of the round surface perfectly. The stirct line will meet itself across the surface of a ball an infinite number of times. The stirct lines can have no deviations from this infinite cycle. A stirct line is the same as a straight line in all things except that it does not have to exist on a plane and that all of its points will meet.

Lee was content to talk about great circles as we developed the theory of spherical geometry in the class discussion. He was, however, reluctant to use "straight" in that discussion. In his journal I continued to ask him to develop and refine his definition of "stirct" by considering lines on a cone, a cylinder, and the walls of our classroom. At the end of the semester, he was still developing these ideas, but had made his definition of "stirct" less restrictive.

Metaphors for Mathematics

In this paper I have described three mathematical situations (the handshake question, the fraction paper, and the Fibonacci sequence and rabbit problem) that help students to reclaim their intuition in mathematics and two mathematical situations (the belted earth and straight on a ball) that help students to move beyond their intuition and use theory, logic, and data to support their ideas. I have discussed how I participate in their developing thinking as I read and respond to their writing in their journals.

I see the signs of change in their view of mathematics as I read their journals, but I am not always prepared for the descriptions of empowerment that are documented in their change papers. The excerpts presented here in the words Janet and Lee used to develop their change papers are typical descriptions of the empowerment students feel through their changing conception of mathematics.

For me math is most like the Sahara Desert. I wander about aimlessly, trying to find the right direction, yet always being fooled by mirages.

I have one other indicator of students' changing mathematical views during this course. Early in the semester I gather the students' metaphors for mathematics. I usually wait until the second week, because a trust relationship needs to begin to develop before the students will be candid about their views of mathematics.

> *Metaphor.* In the classroom setting each student writes his or her own responses to each of the following orally given directives:
>
> > List all the words you would use to describe mathematics. (5–7 minutes)
> >
> > Imagine yourself in a situation of doing mathematics and list all of the feelings that come to mind. (5–7 minutes)
> >
> > List all of the objects (nouns, things) that math is like for you. (5–7 minutes)
> >
> > Read over your three lists and write a paragraph beginning, "For me math is most like a . . ." (20 minutes)

On the first page of this paper are excerpts from the paragraphs written by Writing Seminar students. I collect this first set of metaphors; then, about the twelfth week (of a fourteen week semester), I gather metaphors a second time. The following class I return both metaphors to the individual students and ask them to write the following journal entry reflecting on their metaphors and the similarities and differences they observe about themselves as they read the metaphors they have chosen for mathematics.

> *Journal Entry.* Place the two metaphors that you have written this semester in your journal. How are they similar? How are they different? How does each one of them fit your present view of mathematics and of yourself as a learner of mathematics?
>
> Is there a way you would like to view mathematics that is not yours yet? What metaphor would you choose to describe this new view for yourself? What experiences would you need to be able to adopt this new view?

Janet's Changing Metaphors

To conclude, let me share the metaphors that Janet used to describe mathematics as the semester progressed. Her first metaphor has characteristics that I often see. As she wanders aimlessly she is at the mercy of the elements, and a sudden, blinding sandstorm can destroy any progress she has made:

> For me math is most like the Sahara Desert. I wander about aimlessly, trying to find the right direction, yet always being fooled by mirages. Occasionally I might stumble upon a lake filled with ice cold water, but soon enough I get lost again, surrounded by the expanse of desert. The intense heat gives me a headache as I try to remember where I came from and where I'm going. All the landmarks look the same—each cactus, sand dune, and lizard. The wind picks up and I can find myself in a sandstorm, the sand blinding me totally, making me oblivious to everything else. Once it clears all the knowledge I had gained now is lost, for I'm in a different part of the desert, once again unable to find my way to solid land.

In her second metaphor Janet indicates that she often still feels out of control and still experiences illusions, but she is now on a set course and she can experience exhilaration, fun, and challenge:

> For me, math is most like a roller coaster. Math has its ups and downs. Sometimes there are these really big hills that seem to go straight up forever. It's really a slow, arduous climb to the top. Once you get to the top, there can be that exhilarating feeling of flying down the hill at top speed without a care in the world. At other times, getting to the top is only an illusion. The "top" turns out to be level ground with another huge hill ahead. A roller coaster is also a huge loop, a circle with no beginning and no end. The ride can be fun and challenging at times; yet at other times it can be dark and scary.

Janet, in her journal, talked about these two metaphors:

> I guess I feel that I'm no longer totally <u>helpless</u> when it comes to math. I've come to realize that math is not <u>all</u> bad, as the desert metaphor suggests. There are challenging aspects to math, and if you are courageous enough to brave the big hills, you may be rewarded by a good ride down.
>
> Another difference between these two metaphors is that in the Sahara Desert I was all alone, but on a roller coaster there are other people to go through the experience with you. I've discovered that I'm far from alone in my frustration with math. And yet, I've seen glimpses of how some people are fulfilled by the challenges that math offers.
>
> In a desert, you can walk for hours and not know when you'll get where you are going. At least on a roller coaster, there's a set track and it's the ride that's difficult.

Janet describes her ideal metaphor for mathematics as the following:

> —a forest with patches of sunlight but shade from the trees. Everything is different—different kinds of trees, different kinds of leaves, different kinds of animals. You might get lost sometimes, but you're never scared. Even in the darkest part of the forest the leaves will never be so dense that you can't see the sun.

Remember her "lousy simile," "it was like a light shining through the leaves of a dense forest?" (See page 3 of this paper.) This metaphor is not new to her thinking!

Janet realizes that to adopt her ideal forest metaphor she would need to change her attitude:

> Taking a positive attitude towards math—not saying this is going to be hard for me; this is always going to be hard for me. But saying this is a challenge and I'm going to work at this until I get it.

Janet, who began the course talking about doing mathematics as helplessly wandering in a desert, describes how she now sees doing mathematics in the context of her forest metaphor.

> In doing a problem you think of a solution that is correct, but you aren't satisfied with your first solution; you just keep working on it. Being comfortable enough with a concept to want to keep working with it, to find different paths to a different solution. Just getting the right answer is not the ultimate goal if you want to be in the forest. You want to explore things, not just solve things!

Janet was empowered when she developed a procedure to solve the rabbit problem. She now wants, through mathematics, to explore things, to be unsatisfied with her first correct answer, but to enjoy the challenge of looking for alternative paths, methods, and possibly even alternative results.

In the Writing Seminar I use the metaphors to monitor changing attitudes and to help the students acknowledge their own changing conceptions of mathematical knowledge and mathematics learning. In other settings I gather metaphors in one class, read them, and return them the next class, generating a discussion about mathematics and mathematics learning. This discussion can change the classroom environment to one in which students are more open in their thinking and more willing to try their own ideas. See [5] for the metaphors of some secondary school Advanced Placement Calculus students.

Writing opportunities ... have put me in touch with students' conceptions of mathematics ... The candid writing of students continues to surprise, delight distress, sustain, and teach me ...

Writing opportunities, like those described here, have put me in touch with students' conceptions of mathematics and made me aware of their survival strategies for the mathematics classroom Through nontraditional mathematical situations, involving writing, have been able to change students' disempowered patterns and approaches to mathematics learning and help them feel the empow erment that comes through reclaiming their mathematical intuition and testing that intuition against mathematical theory. Cognitive-developmental theory has helped me understand this change process and guided my design of writing opportunities to facilitate this change. The candid writing of students continues to surprise, delight, distress, sustain, and teach me, as I work to empower students in mathematics. I invite the reader of this essay to try these writing opportunities and others of his or her own design. Begin slowly. Read carefully and seriously what your students say. Ask for clarification to help you really understand their thinking. Be prepared to be surprised. Be patient both with your students and with yourself. You will find that both you and your students will become empowered by the experience.

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Section II

How to "Write Mathematics"

This pair of essays discusses the "nuts and bolts" of writing mathematics. The first, "A Source of Help: Duke's Writing Across the Curriculum Program," addresses how mathematics instructors can teach and evaluate student writing. The second, by a long-time mathematics professor, describes his techniques for getting students to write on their graded work.

A Source of Help: Duke's Writing Across the Curriculum Program

We have been discussing the problems that beset students when they are required to write in mathematics classes. What of the problems that will beset the instructors? Must they learn methods for helping students with their writing? Are they to add to their already substantial burden the new and uncomfortable task of teaching writing? Will the new requirements conflict with the main task at hand, teaching mathematics?

Duke University is building a Writing Across the Curriculum Program that supplies the kind of help a mathematics teacher might need. The major difference between the Duke program and others is its reliance on a new methodology for teaching and analyzing writing. This methodology has proven effective in doing consulting work with corporations, law firms, and governmental agencies; it has been effectively taught for six years at the Harvard Law School; and it is now in use in the undergraduate programs at Duke and the University of Chicago. The creators of this theory are Professors Joseph Williams (University of Chicago), Gregory Colomb (Georgia Institute of Technology), and George Gopen (Duke University). The concepts involved apply to all disciplines and can be applied by faculty from all parts of the university.

The methodology differs from all other strategies in the way it forsakes the more traditional perspective of "writer strategy" for the newer perspective of "reader expectation." "Writer strategy" asks "What can the writer think of to say next?" Such an approach probably grew out of the more immediate problem that afflicted the writing course instructor—how to fill several weeks with assignments engaging enough for students to be motivated to write anything at all. "Reader expectation" asks the more pertinent and lasting question, "Is the reader likely to come away from this prose with the precise thought(s) I intended to communicate?" Effective methods for finding answers to that question will help students to write better *every* time they write, whether it be in their writing class, in a math class, or in any of life's many rhetorical tasks.

"Reader expectation theory" was born of the linguistic discovery that readers expect certain components of the *substance* of prose (especially context, action, and emphatic material) to appear in certain well-defined places in the *structure* of prose [5], [6], [7]. Once consciously aware of these structural locations, a writer can know how to make rhetorical choices that maximize the probabilities that a reader will find in the prose precisely what the writer intended the reader to find.

Readers have what we call "reader energy" for the task of reading each different unit of discourse. (A unit of discourse is anything in prose that has a beginning and an end: a phrase, clause, sentence, paragraph, argument, article, book, etc.) Those energies function in a complex simultaneity: one reads a clause while reading its sentence, which is also part of the chapter.... Each of those energies is available for two major tasks: (1) for perceiving structure (how the unit of discourse hangs together); and (2) for perceiving substance (what the unit of discourse was intended to communicate). For the most part, the distribution of this energy is a zero-sum game: Whatever energy is devoted to one of these tasks is thereby not available for the other. For most expository prose, one almost could define bad writing as that which demands a disproportionate amount of reader energy for discovering structure. If a reader is spending most of the available reader energy trying to find out where the syntax of a sentence resolves itself or how this sentence is connected to the sentence that preceded it, that reader can have precious little left for considering what ideas the writer is trying to communicate. On the other hand, if the resolutions and connections appear exactly where the reader expects them to appear, then the reader can devote most of the available reader energy to perceiving the nature of the substantive thought.

Placing information in one structural location instead of another results in subtle but remarkably significant effects. That subtlety requires that we treat in some detail one example for which we know the authorial intentions (because she told us). Please bear with the non-mathematical example.

Compare these two sentences:

> (a) What would be the employee reception accorded the introduction of such an agreement?
>
> (b) How would the employees receive such an agreement?

Putting aside questions of "better" and "worse," we can probably agree that (b) is *easier* to read than (a). What makes that so? At first we might suggest that (b) is shorter than (a); it turns out, however, that the reduced length is a manifestation of improvement, not a cause of improvement. "Omit needless words" is helpful advice only to those who know already which words are needless.

Instead, we would do better to investigate exactly what is going on in the two sentences, what *actions* are taking place. When we seek out the possible action words of the first sentence, we find several candidates: "be," "reception," "accorded," "introduction," and "agreement." Intelligent readers can find good arguments for interpreting any of these as actions; the author of this intended to convey action in only two of them.

When we turn to the (b) sentence, our task is significantly simplified. It is clear to a majority of readers that "receive" is the one and only action happening in this sentence.

Why the great difference? Because readers *expect* to find the action of a sentence in the sentence's verb. That expectation leads us to perceive action in the verb slot unless that perception makes no sense. In sentence (a), "accorded" sounds like an action but makes no sense as an action. When that expectation is foiled, we have to look elsewhere in the sentence to find the action. Unfortunately, readers have no expectations concerning a secondary structural clue. All that remains is a set of highly interpretable semantic clues: "Reception"? "Introduction"? "Agreement"? Perhaps some concept not actually named by a word on the page? We are using our reader energy for hunting through the structure to find something that the writer could have pointed out to us easily by depositing it in the verb slot. Fulfilling the reader expectation (that the action will appear in the verb) greatly increases the probability that the reader will perceive what the writer intended the reader to perceive.

When something is badly written, more than cosmetic grace is at stake; communication itself may falter. As it turns out, the author of sentence (a) complained that (b) was an inaccurate revision of (a), since it omitted the concept of "introduction," which she had intended as a significant action. New solution: If "reception" and "introduction" are both actions, make them both into verbs. Here is what the author had intended to say:

> (c) How would the employees receive such a proposal if the Council introduced it at this time?

Note that this revision has forced an articulation of *who* is doing the introducing ("the Council") as well as a qualification ("at this time") that makes the action worth considering here.

Revision (c), according to the author, articulates clearly what her intentions had been. Revision (b) may have been brisker and easier to read than (a), but it failed to retain the author's intentions. The problem of the prose in (a) had not been merely a lack of grace, but rather a lack of clarity. We would argue that the two cannot be separated.

Reader expectation theory allows readers to identify a lack of clarity by perceiving a difficulty in structure. We may not know what is missing from the thought, but we can learn how to ask the right structural questions ("What *did* you intend the action of this to be?" "What verb would articulate that?") that will eventually turn up the answers if the author is present. These principles, then, do not give the instructor the power to revise students' prose effectively, but rather allow the instructor to help students revise their own prose. Only the author knows what the author intended.

The same kinds of discoveries concerning reader expectations have been made on the sentence level for the locations of context ("where am I coming from?") and of emphasis ("what is new and important here?"). Yet others have been discovered for the linking material between sentences, for the placement of points in paragraphs, and for the placement and development of thesis statements in complete essays. None of this material is strikingly new in and of itself; good writers, upon hearing the principles, will nod and say they "knew" that, although they had never heard it put quite that way before. The newness of the methodology lies in its having achieved two things: (1) Principles that have been mostly intuitive to this point are now objectified and made conscious for the writer, the better to be controlled and used; and (2) there is now a systematized language with which to speak of reader expectations, no matter what the nature or the field of the substantive material.

These principles have the potential to revolutionize writing instruction. They can be taught to teachers in workshops that last no more than 12–15 hours. They can be used immediately and with wide-ranging effectiveness. They can be taught to students who in turn can use them in evaluating each other's prose, so that peer commenting and grading of writing can become one of the most useful of teaching strategies. Most importantly, the principles allow an instructor to comment not simply on what a student has done "wrong" in a given paper, but rather on what ineffective rhetorical choices a student tends to make with great consistency. The student who puts the action elsewhere than in the verbs throughout one paragraph is highly likely to do so in other paragraphs. Teaching such a student about verb/action reader expectations will aid that student not only to revise the present paper effectively, but also to avoid that structural pitfall in all future writing tasks.

As the product of such revision is by no means merely cosmetic, so the process of that revision is by no means merely mechanical. In order to "fix" a sentence whose action does not appear in the verb slot, a writer has to ask the salient and substantive question "What is going on in this sentence?" The writing process, including this kind of revision process, does not merely lead back into the thinking process; it *is* a thinking process. Eventually the methodology transforms itself from a set of revision tactics to a set of invention procedures. Knowing how structures need to be built eventually leads the writer to recognize logical progressions while still engaged in the original writing process. The result is sometimes a quicker pace of writing, usually a greatly reduced need for revision, and almost always a clearer, more forceful product.

A note about what this methodology does *not* do: It does not propose a new set of rules which must be slavishly followed. It only makes a writer aware of the expectations that most readers have most of the time; the writer can then choose to fulfill those expectations or to foil them. Every one of these reader expectations can be violated to good effect. In fact, the greatest of stylists turn out to be the best violators. (This can be done only if the reader expectations are regularly fulfilled, so that the violation comes as a surprising exception.)

Knowledge of these reader expectations, therefore, should not be used to establish a new set of "rules" akin to our grammatical requirements for coherence (e.g., "use singular subjects only with singular verbs") or our grammatical conventions (e.g., "never split your infinitive"). We cannot and do not intend to argue that the action of sentences *must* be articulated by its verbs; sentences that have their actions elsewhere (or nowhere) abound in the published prose of all fields. Those sentences are not impossible to interpret; they are only less likely to be interpreted by a great many readers in the manner intended by the author. We speak, then, not of rules but of predictions of reader expectations. As a result, we do not list here all the major reader expectations; together they would take far more space to explicate than is here available. A full-length book on the subject is forthcoming.

Duke requires all freshmen to take one of our small (10–13 students) University Writing Courses in their first semester. They are taught the full sweep of the methodology, and they are given opportunities to teach it to each other through a series of peer evaluations. We now have an entire undergraduate student body able to talk the same language about language. Moreover, well over half of our faculty have attended the workshops. As a result, teachers and students in every department are now able to communicate directly, concisely, and effectively about the strengths and problems of students' prose.

The Methodology at Work: Solving Rhetorical Problems in Mathematics

We return now to examples from the earliest lab reports. The rhetorical problems we address here are manifestations of the student-centered problems described earlier, but they do not match up in one-to-one correspondence. The methodology as a whole will affect the student's thinking process as a whole, with the result that both kinds of problems will be substantially reduced.

The list-of-facts paragraph. Perhaps the single most common writing problem in the lab reports resulted from students merely listing their observations, in pseudo-paragraph form. That is, they would indent the first line (therefore "paragraph"), but fail to construct the connections between sentences that would create a coherent unit of discourse (therefore "pseudo").

Reader expectation theory offers several ways of combatting this problem. One can begin by checking the verbs in each sentence. If they consistently fail to state the action of the sentence, it is highly probable that the burden of making the decisions concerning both cohesion and coherence will have been shifted from the writer to the reader.

In the following example we have underscored each of the verbs.

> We <u>observed</u> that the function h <u>is</u> the derivative of the function g, which, in turn, <u>is</u> the derivative of the function f, therefore the function h <u>is</u> the second derivative of the function f. MicroCalc <u>confirms</u> these observations in the Derivatives unit of Quarter 1. Our second <u>observation</u>, concerning the f-g-h relation, <u>was</u> that the relative maxima and minima of an equation <u>are</u> equal to the x-values of the point where the derivative of that equation <u>crosses</u> the x-axis. MicroCalc <u>confirmed</u> this observation also, in the Extrema unit of Quarter 1. The final observation <u>reveals</u> the connection between the f-g-h relationships and questions 3, 4, 5 and 6 <u>answered</u> below.

Note how (with one exception) they all are verbs of observing (on our part), existence (on the part of our observations), or confirmation (on the part of MicroCalc). What does this paragraph tell us?—that the students observed certain things, and that MicroCalc confirmed their observations. It does not convince us that the students have *conceptualized* anything that they have managed to observe.

The purpose of assigning writing in this course was to force students to think about that which they might otherwise perform in only a mechanical way. The constant presence of weak verbs allows the mechanization to continue, the prose acting not as an explanation of thoughts (exposition) but merely as a recitation of facts (narration). Requiring the student to articulate the action of the sentence in the meaning of the verb will force the student to consider what actually is happening during the mechanized process. That, in turn, transforms what was only narrative prose into expository prose. For example, there is one verb in this example that actually does articulate the action of its sentence: "crosses." If the verbs in the surrounding sentences were made to articulate appropriate actions, then the student would be likely to recognize not merely the existence of the derivative crossing the x-axis, but also its significance.

The overpacked sentence with incredible connections.

> We made a table of values for x of $[0,20]$ and narrowed down the interval to $[15,20]$ as well as observing, from Table 1, that at $x=0$, $f(x)$ also equals 0.

In Lab 1, the function f was a polynomial with non-zero constant term; thus, the final assertion in this example is false. However, the connections in the sentence are deceptive, not actual, and thereby distract the reader from noticing that the conclusion is incorrect.

The author of this example could just as easily have continued to add phrases and clauses. No need to stop there: "... $f(x)$ also equals 0, as well as..., after which we..., which resulted in... ." It all would make perfect sense to the writer, who was proceeding linearly through time. However, it would become an increasingly annoying and confusing burden to the reader. How can we tell our students that they have packed too much into one sentence?

Again we can turn to reader expectation theory for help. Readers expect the single most important thing in the sentence—the thing that the writer intends the reader to emphasize—to appear at the end. We call this location the "stress position." Stress positions are created by syntactic closure. Hence every sentence has a stress position at its end, as the period brings it to a close. Colons and semicolons are strong enough acts of closure to create secondary stress positions in the middle of sentences. A sentence is too long when there is more than one viable candidate for any stress position.

In the above example, it is unclear just how many things the writer wishes us to emphasize. As a result, we also do not know precisely the relative weights and the explicit connections we should be observing. Do either of the following express more accurately the authorial intent?

> In making a table of values for x of $[0,20]$, we narrowed the interval to $[15,20]$.—[Reader's thought: What's the connection?]—From this table we observed that at $x=0$, $f(x)$ also equals 0.

> We chose to limit the table of values for x to $[0,20]$ because... . Similarly, we narrowed the interval to $[15,20]$ because... .—[What's the connection?]—We observed from the first table that at $x=0$, $f(x)$ also equals 0.

Given the author's prose, we now see that we are unable to perceive the authorial intent. Notice how much thought and information we lack to make the proper balances and connections. Insisting on one emphatic point per stress position will unearth (or stimulate) a great deal of student thought. It will also make it clearer to students when they yet have no point to make.

Lack of agency: Nobody did anything; things just happen; things just are.

> The function g is the derivative of the function f. The function h is a derivative of the function g. This is displayed on the table marked C. ...

Note how nobody *does* anything here. Students were asked to report what *they* saw and did. and yet they are nowhere to be found in the prose. Some will complain that they were carefully taught in high school not to write in the first person. Reader expectation theory demonstrates that the herring of first person usage is a red one. It matters not whether one ought or ought not to use the first person; it matters only whose story is being told. Readers firmly expect that the story of a clause or sentence is about whoever appears at its beginning. If a lab report must tell what students observed, then students ought to appear quite often at the beginning of clauses and sentences.

> We discovered that the function g is the derivative of the function f, because.... Using the same principle, we also perceived that the function h is a derivative of the function g. (See Table C.)

Combining the principles of action-in-verb and whose-story-up-front, students can lead themselves to discovering the emptiness of their own statements. We were surprised, for example, by the frequency with which students wrote that they "received" answers. That formula—"We received the answer of 3.742 for x"—reveals and sums up for us the central pedagogical problem in calculus courses.

In sentences written in the active mode, the person whose story it is also turns out to be the agent of the action. Students find they can avoid some of the consequences of making mistakes if they substitute other agencies for their own at the beginnings of sentences. For example (from Lab 2, devoted to turning points):

> The complexities of the graph of $f(x) = (x^4 + x - 1)/(x^3 + 2x^2 - 1)$ resulted in us not being able to correctly determine any turning points on the graph of that function.

Whose fault? "The complexities'."

Take a second look at an example we cited earlier:

> Any difficulties in the lab occurred because some numbers were not quite accurate. The computer only carried out the figures to the sixth decimal place, and the students tried to make the results as accurate as possible by narrowing the range, a and b on the table of values while at the same time increasing the number of intervals, N to up to 500.

In the first sentence, "difficulties" occurred. In the second, "the computer" fails badly to do its job, but "the students" come to the rescue ingeniously. If the students had been active earlier in the prose, they might have come to understand that for the approximate results required, a much shorter table (20 numbers) would have sufficed. By escaping responsibility here, they landed themselves with several times the labor.

The multi-topic paragraph. Here is yet another fact-filled, chronologically linear paragraph.

> From our previous study of calculus, we determined that $G(x)$ is a first derivative of $F(x)$, and $H(x)$ is a second derivative of $F(x)$ and a first derivative of $G(x)$. We confirmed our assumption by using the 'Derivative' command from MicroCalc. We took the first two derivatives of $F(x)$, and they were the same as $G(x)$ and $H(x)$, respectively. Next, we found the first derivative of $G(x)$, which resulted in the function $H(x)$. We also observed that each time $F(x)$ would reach an extreme point

> $G(x)$ would be equal to zero. This held true for $G(x)$ and $H(x)$. When $G(x)$ reached an extreme, $H(x)$ also equalled zero. We confirmed this by using the 'Extrema' command on MicroCalc. Using this command we found out the extreme values for $F(x)$. Then, we plugged those x-values into $G(x)$, and $G(x)$ was found to be zero.

In addition to all the revision tactics suggested above, we can approach this example in terms of reader expectations of paragraph structure. Readers expect an expository or argumentative paragraph to be about one main point. That point should normally be expressed in a single sentence, from which or to which all other sentences flow. This paragraph violates reader expectations significantly:

1) There are too many points being made for the number of stress positions available;

2) No one sentence articulates the point that gives shape to the rest of the paragraph.

We have found, however, that students tend to have few problems with the concept of the *unity* of the paragraph. Indeed, they seem delighted to bring their momentary labors to closure by ending the topic and indenting the next line. How, then, can the above example of prose have been written as a paragraph? The answer we suggest to our own question we find most disturbing. To the student, the entire paragraph was indeed on one topic—the most important topic of all: "How much can I say about this problem so that I can get full credit for the answer?" All the sentences in the paragraph connect to that (unstated) point sentence.

We need to move our students away from being centrally concerned with producing enough material and direct them towards grasping and developing concepts. We can help accomplish this by getting them to understand the reader expectations of paragraph structure. They will be hard pressed to fulfill those expectations without entering into the kind of conceptualizing thought process that we wish them to experience.

You Can and Should Get Your Students to Write in Sentences

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For nearly two decades I have insisted that the students in mathematics courses that I teach write in complete English sentences on all written work that I grade. I do this in *every* course; not just in courses for majors in mathematics or courses in which writing is emphasized. While it is rare that I teach courses below the level of the calculus, I do this in such courses as well. I do not go as far as to insist on good paragraph structure or style, but I never read, much less grade work not written in sentences. I point out that symbols are used to abbreviate words or phrases and may be parts of a properly constructed sentence. Thus, "$x = 2$" is a sentence with subject "x," verb "$=$," and object "2" while "$2x = 4 = x = 2$" is not. I insist, also, that any symbols introduced by the writer whose meaning is not clear from the context be defined or quantified properly.

You do not have to teach your students how to write or recognize a simple English sentence unless some of them are foreigners with a serious language deficiency, in which case they should not be in an ordinary college class, and some sort of extraordinary help will be needed. What I am saying does not apply to such students, but does apply to almost all others, independent of their mathematical talents. They can write in sentences, but typically, they will not unless you *really* insist!

On the first day of class I spend some time explaining what the course is all about, what my policy is on homework, quizzes, midterm exams, the final exam, and how I will determine their final grade. I indicate at that time that I am more interested in how they arrive at answers to problems than I am in the answer itself (which often may be found in the back of the textbook), and that I will not read their work unless it is explained *with the aid of complete English sentences.* I stress that I will read what they write and will not try to guess at what they really mean, and that I do not separate form and content. I tell them that I will not grade their English; I just won't read work written in a private code. To reinforce this, I summarize it in a two or three page handout including examples of satisfactory and unsatisfactory solutions of problems which contain explanations of why they are unsatisfactory. One that I have used for a linear algebra course required of all students at Harvey Mudd College appears below.

The initial reaction of most of the students is to ignore both my talk on writing and the handout. They are identified with all the other copy book maxims of education and exhortations given by their teachers about which they need not do anything. Sentences are for courses in English and papers written for courses in history or economics or similar subjects; surely they have nothing to do with mathematics. I collect and grade some written work early in the semester (usually in the form of a 20 minute quiz). Students are shocked when they read comments such as: "I cannot follow this," or "Where is the explanation?" or "This is not a sentence." followed often by the phrase "Not read further." When these comments are accompanied by large losses of credit, they begin to take my words and the handout as something with which they must cope. I ask all who have done poorly to come to my office for a conference and many others come to talk with me as well.

Usually they admit to not having read the handout. Some do not remember receiving it and others say that it was misplaced, so it pays to keep some spare copies on hand. Initial reactions range from incredulity to anger. Some sample encounters follow.

Comment: Dr. Henriksen, nobody ever asked me to write this way before.

Reply: That's too bad. The time to start writing clearly is now. How can I help you?

Query: I am sorry that you had trouble reading my explanations. Is it all right for me to go over them with you now?

Reply: Since you were told that you were expected to explain your work in writing, I cannot accept an oral explanation as a substitute and restore the credit you lost. On the other hand, I will be only too glad to tell you if your current idea of an explanation is a good one, and to help you find better ones.

Assertion: I got the right answer and anybody can understand what I am doing! (Usually said with irritation if not blatant anger.)

Reply: I am a professional mathematician and I can't understand what you have written. I see hardly any words, much less sentences. There are lots of symbols used without telling the reader what they denote. Would you like to go over some ways in which you might have explained your work clearly?

Assertion: I demand that you tell me how many points you are deducting for English and how many for mathematics! (Said with great anger.)

Reply: Since I could not understand your work, I cannot answer your question. Why don't you try to explain to me orally what you were doing and we can explore ways to express it in writing and see if it is correct.

The idea is to remain calm in the face of obvious attempts by students to remove your pants or to intimidate you with anger, and turn them in the direction of learning how to explain their work by writing in simple sentences. I did not always succeed in keeping cool. Once, while on a visit to another college and teaching first semester calculus, a student came to my office and proceeded to haggle over my grading of each question on the first hour exam. When I directed him to the handout, he told me that I would not be very popular if I kept on doing things like that. I could not hide my anger when I told him that I would penalize him even more in the future for poor explanations, and I would take care of my own problems with popularity. Thus, with this very bright (albeit insolent) student, I lost the chance to turn this into a constructive experience.

On the next quiz more students wrote in sentences, and there was a slow but steady improvement in the work of the students. Many resent having to change their ways, but bit by bit they fall into line, and it becomes much easier to read and understand their work. You have to stand fast as they test your determination repeatedly. Once they realize that you will not waiver, they start asking questions about what is a satisfactory mode of presentation, and the battle is won. It can take anywhere from half to two-thirds of a semester, but unless a student drops the course in anger, he or she does get in the habit of writing in sentences.

Those most likely to get angry at demands that they explain their work and write in sentences are at the extremes of mathematical ability. Some very bright students are used to "seeing" how to solve many problems in a *gestalt* without having to examine exactly what they are doing. I try to point out that this will not be possible forever and, hence, that they must learn how to explain what they are doing. It is particularly important that I convince such students that my writing demands are more than exercises in pedanticism. At the opposite extreme are the students who gave up on trying to understand mathematics. They had acquired certain skills in the past in a *gestalt,* and then moved to memorization when that no longer worked. Such students are very insecure and resent what they regard as yet another demand. When you can get them to listen, it is important to point out that good writing is a reflection of clear thinking, and clear thinking rather than memorization is the key to success in mathematics. I never accelerate my demands or spend class time haranguing my students on the virtues of good writing. It seems to be enough to go over briefly what it means to explain work and write in sentences, while never compromising in the face of complaints or abuse.

Good writing is a reflection of clear thinking, and clear thinking rather than memorization is the key to success in mathematics.

By now many who have been patient enough to read this far must be asking why it is worth accepting so much *sturm und drang* to attain such a modest goal? Why not spend the time needed to teach them how to write mathematics really well and assign a term paper? The answer is that any required course that I teach has such a crowded syllabus that I cannot spend the time needed to teach them how to write at such a high level without leaving out some of the topics our department is obligated to cover. It is difficult also to teach students how to write well if they regard writing in sentences as the equivalent of dressing up and going to church at Easter time. They can write in sentences, and doing so helps them break the habit of doing mathematics by throwing some symbols on a page and stirring them in hope of getting the "right" answer.

Some years ago a Pomona College senior enrolled in a pre-calculus course I was teaching that was normally taken by freshmen. He was a bright underachiever who had put off a mathematics requirement as long as he could. He ignored my handout and my request that he write in sentences over and over again. I called him into my office several times and asked him why he persisted in paying no attention to instructions, and said that I could not help him as long as his work consisted only of disconnected batches of symbols. When I discovered that his major was English, I lost patience and almost yelled at him, "You know what a sentence is! Why do you *refuse* to write in sentences?" I handed in a report of unsatisfactory work at mid-semester and I hoped secretly that he would drop the course and go away. Suddenly he began to explain his work and write clearly! By the end of the semester he had earned a grade of B and probably would have done better if it had lasted longer. He explained his long delay in trying to write anything comprehensible on his mathematics papers to his inability to associate it with anything but rote memorization. All that he had learned in the past had been on a *monkey-see-monkey-do* basis. This experience convinced me that the best favor I could do for my students was to insist they explain their work with the aid of complete English sentences!

This program always succeeds in that all of the students are writing in sentences well before the end of the semester, which makes it much easier to read and grade their work. In recent years I have included "answer only" questions on quizzes and hour exams, for which no partial credit is given and no work is read, to help convince students who think that they know the material, but get lower grades than they deserve because of my unreasonable demands, to realize that they are deluding themselves. While this helps a little, those that find that I will not choose among multiple answers to a single question or guess at the meaning of an ambiguous answer get just as upset as those who object to having to write in sentences. The upset might be less if similar demands were made in their other courses.

I have given up on trying to convince my colleagues to adopt this program. Some feel it unfair to demand that students write in sentences on examinations if their homework is not graded in the same way. Since homework is graded by undergraduates who are little better than answer checkers, and who themselves write poorly, this latter cannot be done. (I wonder if my colleagues would use this as an excuse if there were no students to grade homework.) Others plead a lack of time to engage in such a program, despite my offer to help and my claim that it saves time. I suspect that the real reason for the lack of willingness to insist that students explain their work is fear of complaints in an era in which popularity is equated with good teaching by so many college administrators.

In no way does this program take the place of the many much more serious attempts to teach students how to write substantial mathematical tomes. Indeed, when I teach advanced courses to students majoring in mathematics, a lot of my efforts are concerned with getting students to write in paragraphs. On those occasions when I wish to collect a term paper, I have to spend even more time teaching students how to write. The virtues of the program described above are that it is simpler to carry out, it reaches a wider audience, and it always seems to work.

I conclude with a sample handout on linear algebra from which the reader may construct one suitable for other courses.

Handout on Linear Algebra

Your grade in this course will depend almost exclusively on written work (homework, short quizzes, and examinations), so it is important that you learn how to communicate clearly in writing. Any work you submit for evaluation calls for an explanation of what you have done with the aid of complete, grammatically correct English sentences. (Symbols abbreviate English words or phrases and may be used as parts of sentences.) I will read exactly what you have written, and will make no attempt to deduce what you "really" mean or to supply missing steps or logical connectives. Any symbols you introduce that are not standard must also be explained or quantified. Make sure, also, that you supply an explicit answer to each problem you claim to solve.

In particular, I do not separate form from content. If I can't understand some part of your work, I will not struggle to read it, and your grade will suffer accordingly; even if you got the "right" answer.

Your explanations need not be lengthy to be clear. I conclude with some examples.

PROBLEM 1.

Find all vectors (x, y) in the plane such that $x + y = 0$ and $-3x + 2y = 1$.

Unsatisfactory Solution

$x + y = 0$

$-3x + 2y = 1$

$+3y+2y=1$

$y=1/5, x=-1/5$

No explanation has been offered and the question has not been answered explicitly.

Satisfactory Solution

(1) $x+y=0$

(2) $-3x+2y=1$

By (1), $x=-y$, whence by (2) $3y+2y=1$. So $y=1/5$ and $x=-y=-1/5$. Thus the only such vector is $(x,y)=(-1/5,1/5)$.

Problem 2.

Suppose $A=\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ -5 & 1 & 3 \end{bmatrix}$.

Evaluate the determinant $det\,A$.

Unsatisfactory Solution

$det\,A=6-40-30-4=-68$

The work has not been explained and there is no way for the reader to check its accuracy.

Satisfactory Solution

By the diagonal rule,

$$\begin{aligned} det\,A &= 1\cdot2\cdot3+2\cdot4\cdot(-5)-3\cdot0\cdot1-(-5)(2)(-3) \\ &\quad -1\cdot4\cdot1-3\cdot0\cdot2 \\ &= 6-40-30-4 \\ &= -68. \end{aligned}$$

Problem 3.

State whether or not each of the following sets of functions is a subspace of the vector space V of all real-valued functions of a real variable.

(a) $S=\{y\in V : y''+y=0\}$.

(b) $T=\{y\in V : y''+y=x\}$.

Unsatisfactory Solution

(a) $(af+bg)''+(af+bg)=a(f''+f)+b(g''+g)=0$.

Yes.

(b) $(2x)''+2x=2x\neq x$.

No.

Symbols are used without definition. The work is cryptic. Neither question is answered explicitly.

Satisfactory Solution

(a) Suppose f and g are in S, and a and b are real numbers. By the sum and product rules for derivatives,

$$\begin{aligned} (af+bg)''+(af+bg) &= a(f''+f)+b(g''+g) \\ &= a\cdot0+b\cdot0=0. \end{aligned}$$

Hence S is a subspace of V.

(b) Let $h(x)=x$. Then

$$h''(x)+h(x)=0+x=x,$$

so h is in T. But

$$2h''(x)+2h(x)=2x\neq x,$$

so $2h$ is not in T. Thus T is not a subspace of V. [Note that a subset K of V is a subspace of V if and only if $a,b\in R$, and $f,g\in K$ imply that $(af+bg)\in K$]

Section III

Writing Assignments

Suggested assignments for mathematics classes, some specifically targeted at core mathematics courses and some generally usable at any level, follow. The four essays also include tips on timing of assignments, grading, and feedback both to and from students. The list of general exercises is not comprehensive, nor is it intended to be. These exercises are meant to be adapted to the needs and desires of the instructor using them. The most important factor is feedback. Remember, cadets aren't necessarily used to writing in their mathematics classes. To learn how to write a good answer, they'll need to see good answers as well as bad ones, and they'll need you to tell them what is good or bad about each.

Library and Writing Assignments in an Introductory Calculus Class

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During the last several years the mathematics community has engaged in a great deal of discussion concerning the teaching of calculus (e.g., the MAA conference reports, *Toward a Lean and Lively Calculus* and *Calculus for a New Century*). Over the last twenty or thirty years the emphasis of the calculus course has shifted from understanding concepts to problem solving. Listen, for example, to Lynn Steen, former president of the MAA:

> [Freshmen courses should] help students learn to think clearly, to communicate, to wrestle with complex problems.
>
> There is very, very little in the calculus course of today that does any of these things. Word problems are a small step in that direction, but they are rare ...

Thus while we once taught concepts in the classroom and hoped that, as a by-product, students would learn to solve problems, today we teach problem solving and hope that eventually students will learn the theory behind it all. Unfortunately it rarely works. Now the question is "How do we reverse this trend?" I believe that an answer may come in borrowing an idea from our colleagues in the humanities—if you want students to understand something, have them write about it.

If you want students to understand something, have them write about it.

During the fall of 1988 I taught two sections of introductory calculus in which students were given library and writing assignments. Each student was given two assignments, the first concerning a topic from precalculus and the second on a topic in calculus. They were instructed to research their topics (via either textbooks which I had put on reserve in the library or their own high school or other texts) and then to paraphrase their research in a term paper which was to be written in a clear, concise, readable fashion. I emphasized that it must be technically correct. They were also to write a paragraph critiquing each of the textbooks from which they read.

There are several reasons why I chose their first assignment to be on a topic from precalculus. First, I wanted them to do it early in the semester, which meant that it would be before most of them had been introduced to such calculus topics as limits or derivatives. Second, I wanted them to write their first paper on a topic they felt they already understood, and hence a topic with which they did not feel too uncomfortable.

During the first day of classes I told the students about the two assignments they would be given and briefly described what would be expected of them. At the end of the first week I handed out the following list of topics from which I (not they) would choose their assignments:

1. Completing the Square and the Quadratic formula. Include a short discussion of the method of completing the square, and then show how that method is used to derive the quadratic formula as the solution(s) to the general quadratic equation, $ax^2+bx+c=0$. Also include a discussion of the discriminant and its relation to the roots of a quadratic equation.

2. Inequalities. Discuss the properties of inequalities and the interval notation for solutions to inequalities. Outline a general method for solving quadratic (second degree) inequalities. Be sure to include a discussion of the difficulties involved in multiplying or dividing an inequality by an unknown quantity or expression.

3. Functions. Give the definitions of a relation and a function (give examples to illustrate the difference). Define and discuss the domain and range of a function. Discuss composite and inverse functions.

4. Systems of Equations. Define a linear equation in two unknowns. State and illustrate a general method for solving two linear equations in two unknowns. Discuss why this method might not work for nonlinear equations.

5. Absolute Value. Give the definition of absolute value. Discuss some of the properties of absolute value (e.g., $|ab|=|a|\,|b|$). Also discuss the relationship between absolute value and radicals. Tell how absolute value is related to distance on a number line.

6. Symmetry and Graphing. Give the tests for symmetry with respect to the x-axis, the y-axis, the origin, and the line $y=x$. Discuss the geometric significance of each of these symmetries. Give examples of each.

7. Transformations, Translations, and Stretchings. Discuss vertical and horizontal translations; vertical and horizontal stretchings. Compare the graph of $y=f(x)$ to the graphs of $y=af(x)$, $y=f(ax)$, $y=f(x-b)$ and $y-b=f(x)$ for real numbers a and b.

8. Equations of Lines. Discuss the equations of horizontal and vertical lines, the point-slope form of the equation of a line, and the slope-intercept form. Also give a short discussion of parallel and perpendicular lines.

9. Factoring Trinomials. Discuss your favorite method of factoring a trinomial into the product of two binomials. Also, state the factor theorem and tell how it is related to the problem of factoring a polynomial expression.

10. Slope of a Line. Give the definition of the slope of a line and discuss why it doesn't depend on which two points you choose. Also give a discussion of the interpretation of a large, positive slope; a small, positive slope; a small, negative slope; and a large, negative slope.

11. Rational and Irrational Numbers. Define rational and irrational numbers. Prove that $\sqrt{2}$ is irrational. Which rational numbers have decimal expressions that terminate? Which ones repeat?

12. Synthetic Division. Discuss and illustrate this method of division of polynomials.

A week or so later I made the individual assignments and encouraged them to begin work. The topic assignments were made somewhat at random, although by that time I was beginning to recognize which students were roommates or close friends and I tried to avoid giving them the same topics. Ideally, I would like to have been able to assign a different topic to each student, but obviously there were far more students than topics.

The students were told to write an article of from two to four typewritten pages on their assigned topic. I emphasized that the article should be written in the student's own words, paraphrasing but not copying the texts that they read. I encouraged them to try to remember the problems or frustrations they may have encountered when they were first introduced to the topic in high school, and to write as though they were writing a portion of a textbook with the aim that other students be able to read and understand it. I encouraged them to think of a younger brother or sister who might be studying the topic this year, and to ask themselves what they might write to help that younger sibling gain a deeper insight and understanding of that topic. I also encouraged them to supplement their articles with examples, graphs, or illustrations as long as they were sure to label carefully. Finally I reminded them to write a paragraph reviewing the section of each textbook from which they read. I emphasized that they should discuss what they liked as well as what they disliked about each book.

I encouraged them to think of a younger brother or sister who might be studying the topic this year, and to ask themselves what they might write to help that younger sibling gain a deeper insight and understanding of that topic.

I must admit that at first I was apprehensive about grading their papers. I have taught introductory calculus enough now that I feel very comfortable about making examinations for any and all parts of the course and about assigning grades to the results. However, grading term papers was a whole new experience—one for which I did not feel particularly well prepared. But while I did find myself agonizing over some of the papers (how much original thought was involved here? how much credit should I give there?), by and large the grading went smoothly. While most of the students took the assignment seriously and did a good job, some of the papers obviously had been done the night (perhaps even the hour) before class. For example, four students from the two classes were assigned the topic of factoring trinomials. Three of the four wrote a page or so carefully outlining the "FOIL" method that they had learned in high school, supporting it with two or three examples, then outlined other methods they read about in precalculus books, discussed the more general problem of factoring polynomials, stated the factor theorem, and discussed, via examples, how it is used in factoring. They concluded with one or two paragraphs of criticism of the books they had read. The fourth paper, however, which was about one and a half pages in length, quickly discussed the "FOIL" method (no examples) with several incomplete sentences and even more incomplete thoughts. There was no discussion of the factor theorem and no indication of which, if any, books he had read in researching the assignment. It was not difficult to decide the relative merit of that paper.

Another problem in grading these papers was that the assignments themselves varied in difficulty. For example, the students who were assigned to write about rational and irrational numbers encountered more problems than the others. With a little help they were all able to read and do a reasonable job paraphrasing a proof that $\sqrt{2}$ is irrational; but none had a guess as to which rational numbers have decimal expansions that terminate and which ones have expansions that repeat. At the other end of the spectrum, I only included the topic of synthetic division because I had run out of ideas. I knew in advance that it would be difficult for students to express any cleverness or originality with this topic, and I had to keep that in mind as I graded those papers. I should say, however, that one of the best papers I received was written on that topic. All in all the reading and grading of these papers did take up a fair amount of extra time on my part, but the grading was not as difficult as I had anticipated.

The second assignment was made much later in the course. After about the tenth week of the semester I handed out the following list of topics:

1. Limits. Give the definitions of the limit of a function, one-sided limits, and infinite limits. Give your own pedagogical description of what a limit is and how to find it. Use the definition to give a careful proof of a limit problem (e.g., prove that $lim_{x\to 2}(3x+1)=7$).
2. Continuity. Give the definitions of continuity of a function at a point and on a set. Give your own pedagogical description of the meaning of continuity. Use the definition to prove that the composition of two continuous functions is continuous.
3. Tangent Lines. Give your own pedagogical descriptions of secant lines and tangent lines. Give the definitions of the slope of a secant line and the slope of a tangent line. Show how this leads naturally to the study of the derivative.
4. Instantaneous Velocity. Give your own pedagogical descriptions of average and instantaneous velocity. Then give the definitions of each and show how this leads naturally to the study of the derivative.
5. Derivatives. Give the definition of the derivative. Give several examples. Include a discussion of circumstances under which the derivative fails to exist. State and prove the product rule for derivatives.
6. Relative Extrema. Give the definitions of (relative) maximum and minimum values of a function. Discuss increasing and decreasing functions. Then give a general discussion of how to find maximum and minimum values for a function.
7. Concavity. Define the concepts of a function being concave upward and concave downward. Give a general discussion of how to find where a function is concave upward, concave downward and points of inflection.
8. Chain Rule. Review the definition of the derivative and then state and prove the chain rule. Also give several examples of how it is used.
9. Differentials. Give the definitions of dx and dy. Discuss the difference between Δy and dy. Give examples to show how dy is used to approximate Δy.
10. Mathematical Induction. State the principle of mathematical induction and give your own pedagogical interpretation of its meaning and why it is true. Use it to prove that $1^2+2^2+\cdots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$.
11. The Mean Value Theorem. State and prove Rolle's theorem. State the mean value theorem and give your own pedagogical description of its meaning. Illustrate with examples.
12. Least Upper Bound Axiom. Give the definitions of upper and lower bound for a set of numbers. Give examples of sets which are and are not bounded above. Define least upper bound and greatest lower bound for a set of numbers. Give examples of sets which do (do not) contain their least upper bound. Also state and discuss the Archimedean property of real numbers.

A week or so later I made the individual topic assignments and they were given two weeks to complete their papers. Since by this time we had covered each of these topics in class and they had been tested over that material, I attempted to make topic assignments coincide with weaknesses as observed on their tests; except that again I tried to ensure that roommates, friends, or others who might be tempted to work together were assigned different topics.

In my oral instructions to the students I emphasized that while these (second) papers need not be any longer than the first, I did expect them to research their topics much more thoroughly. I also stressed that although I again wanted them to paraphrase and not copy from the textbooks they read, it was very important that their writing be technically correct. I warned them that this would be graded much more strictly than in the first paper.

Indeed, I did observe significant improvement in the quality of writing in the second assignment. Not only quality, but quantity as well. Most of the papers were quite a bit longer than I had required. The average length was more than six pages. But what I found most interesting was their written evaluations of the textbooks from which they had read. In the first paper the vast majority of the criticisms were positive—some even highly positive. However, in the second assignment many were very harsh in their criticisms of the first book from which they read and progressively less so with the second and third texts. In fact, many had great praise for the last text. Since I had put only eight calculus books on reserve at the library, and since each student selected them more or less at random (and in random order), the same book was severely reproved by some and highly praised by others. In retrospect, I believe that most students had difficulty understanding what they were reading the first time they read it, and held the author of their first text accountable. Then as their comprehension began to increase, the second and third texts miraculously became more readable.

At the end of the semester I asked each student in these introductory courses to write a paragraph giving their reaction to the writing assignments. In general their comments were very positive—some even admitted to having learned something. For example:

> I liked the idea, although I'd never heard of writing a paper in math before. I learned a lot from it. I think it's a good idea.

> . . . it helped us understand the material better—and it wasn't as if they were big, huge term papers as in English and such.

> At first I thought a term paper for calculus was a stupid idea. It's still not my favorite part of the class, but it did help a bit . . .

> When I first heard about term papers in a math course, I felt that it was first impossible and second pointless. However, the papers I wrote have given me a much better understanding of the concepts we studied. Up until this course, I did my math by memorizing formulas and equations without really knowing what was actually going on. Now that I've had to explain a formula . . . I've really learned the reasons behind the equations. In fact, I've told my high school teacher about this idea and he's seriously considering it.

I must confess, however that not all the comments were positive. Witness the following:

> I think that the papers were unnecessary and that the students' time should be used to solve calculus problems instead. I did learn from the papers I wrote, but I don't think it made much difference in learning to solve calculus problems.

This project was not intended to have measurable short-range success. In fact, the final grade point averages from these two classes are not significantly different from other introductory calculus classes I have taught. This is not surprising to me. The purpose of the assignments was to encourage students to think more deeply about certain concepts; the written tests still emphasized problem solving. The real test of the effectiveness of this project will come when and if these students decide to become mathematics majors and take our upper-division courses. Hopefully, their transition from the calculus and other lower-division courses will be much smoother than is the experience of most of our students. However, in order that this smooth transition become a reality, I believe it is important that we begin teaching students to write mathematics at all levels of the curriculum for the major. These types of writing assignments can be included in all levels of calculus courses, not just the introductory course. Also, they could be implemented in such sophomore courses as linear algebra, discrete mathematics, and differential equations.

Using Writing to Improve Student Learning of Statistics

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Let me tell you about the experience that first showed me the need for student writing in applied statistics. I had set an examination question that required my students to do a hypothesis test. It ended with a poorly worded question that students interpreted in a variety of ways. Some simply provided the results of their calculations along with a number they had extracted from a statistical table. Others included some jargon about "rejecting the null hypothesis" while others stated a conclusion in more practical terms, such as "the tested drug is probably more effective than the standard treatment." Some students provided two or even all three of these responses. Of course, all three constitute restatements of a single fact in different language. Unfortunately, I found little or no correlation between the different answers of students who gave multiple answers. If the numbers clearly indicated that the null hypothesis should be rejected or the treatment declared effective, students were just as likely to say the opposite, even when their computations were perfectly correct.

Reflecting on my students' answers, I reached a number of conclusions.

1. Since their final conclusions were no better than what they might have reached via a simple coin toss, all the complex computations I had taught them were of no real value.

2. My students' lack of understanding was mostly no fault of their own. Their textbook spent pages and pages showing them worked examples of how to do the computations, but far less space discussing what the computations meant. Exercises asked them to perform computations, but rarely asked them to explain their results. Rarely were they required to *select* an appropriate technique. The appropriate technique was always whatever technique was described last.

This led to some serious thought about what my students needed to learn in a statistics course, and how I might help them to learn those things.

I next asked myself what my students were likely to need to do with statistics after graduation. I tried to order these needs on the basis of how many of my students might have them. I hope you will pardon my listing those needs here, because they are relevant to all kinds of "book learning."

1. Virtually all of my students would need to evaluate quantitative information presented to them in newspapers, at zoning board meetings, by their doctor, or by numerous other sources. These students need to know what a median or a standard deviation is or means. They need to know the strengths and weaknesses of these numbers as summaries. They need a healthy scepticism toward quantitative claims.

2. A smaller group of my students would need to evaluate the meaning and propriety of more technical statistical techniques that might be used by researchers in their own field.

3. A still smaller group of my students might need to evaluate statistical work done by subordinates or provided by consultants.

4. A very small group of my students might actually carry out a statistical study themselves. They would almost certainly use a computer to carry out the mechanics of data storage, editing, and analysis.

5. An even smaller number of my students might one day need to carry out a large scale statistical study while stranded on a desert island, or at a remote wilderness location, or in some other situation in which a computer would be unavailable. These students would need to know how to perform the computations by hand.

If we look at most statistics books, and most statistics courses, we find them organized as if my last group of students were the norm. Indeed, the whole pyramid is inverted. Few textbook problems deal with meaning or interpretation, something *everyone* needs to know about. Instead, most deal with computational techniques that few students will ever use outside the classroom.

So, I resolved to try to spend more time on meaning, evaluation, and interpretation. However, my new found idealism was tempered by a basic fact of schooling: the students won't learn anything that does not appear on the exams. The simple conclusion is that questions involving meaning, evaluation, and interpretation must appear on the exams. Once we reach this conclusion, the need for writing is obvious: *the answers to questions of meaning, evaluation, and interpretation are verbal, not numeric.* Thus writing becomes, not just another subject to teach, nor even a tool for achieving traditional goals, but rather a necessary path to developing higher level quantitative skills.

These, then, are the values and experiences that have shaped my interest in Writing Across the Curriculum. Let me now deal with some of the practical problems of implementation. The most important piece of advice is: start slow. Your students have had an average of 14 years of experience with teachers who preached the importance of higher level skills but tested only on memorization and manipulative skills. Your best sermons will therefore have no effect, and your students will all fail that first exam when you ask them questions exercising skills they have never developed. You will become discouraged, curse their stupidity and your own idealism, and return to rote drill. Actually your students can do far more than you imagine, but they need your help. There follows some advice on providing that help. Bear in mind that it is based on all of the above. If your reasons for using writing assignments differ from mine, you may prefer a different approach.

The first thing you need to change is your teaching. Deemphasize mechanics. Assign only enough computational problems to get the ideas across. Keep the numbers *very* simple. Encourage the use of calculators or computers for any computations beyond the bare minimum needed to grasp the concepts. Spend lots of class time on interpretation and meaning. *Provide sample test questions*! These communicate the nature of your expectations and the fact that you are not kidding. Once you have taught the course this way a few times, you will have a bank of old exams. Share them freely. Let students see for themselves that you really do ask embarrassing questions on exams. Distribute these old exams well in advance. Students can not change their study habits the night before an exam. Indeed, you will find that they will initially, but very strongly, resist changing their study habits at all. There really is not much you can do about that except to fail those who do not perform at the level you desire. Things will improve as word gets around and students enter your class with expectations already tempered by your reputation.

Then there is the matter of writing exam questions. Start small. Problem 1 on Exam 1 should not be

> Compare and contrast the methods, assumptions, uses, and histories of parametric and nonparametric statistical techniques, giving special attention to their impact on the methodology of the social sciences.

A more reasonable start might be

> For the data 3, 1, 4, 1, 21, find the mean, mode, and median. Which of these would best summarize this data? Why?

Since many of my readers may not teach statistics, I do not want to give further statistical examples here. The principles should be clear.

Keep in mind that the main goal is to force the students to *think*. Forcing them to write is just a tool, a way to hold them accountable for thought. You do not have to make them write a lot of words as long as you get them to think a lot of thoughts. One-sentence answers may meet your goals. Also keep in mind that reading and writing may often be interchanged. Instead of asking students

> Find the slope in $y = 2x + 3$.

Or even

> Interpret the slope in $y = 2x + 3$.

You might ask

> How much does y change for a unit increase in x when $y = 2x + 3$?

Now the answer is a single number—much easier to grade than a student-written sentence or paragraph on the subject.

Sometimes teachers are discouraged by the quality of writing they get, or discouraged from asking for writing by fears of what they *might* get. In my experience, lack of mastery of subject matter will far outweigh any writing flaws. Indeed, you may discover that your students know far less than you thought about the meaning of those numbers you taught them to calculate. This can be taken as a sign of either the futility or the importance of your work, depending on your outlook on life. You should work on teaching your discipline until the content of the answers is better than the expression. In the process, you will find that the expression improves by itself. No one communicates well when they have not the faintest idea what they are talking about.

Sometimes teachers are discouraged by the quality of writing they get, or discouraged from asking for writing by fears of what they *might* get.

Yet another issue is grading student writing. Here my solution is as simple as it is radical: *don't*. I grade students only on such knowledge of statistics as they are able to communicate to me. As long as their mastery of the mechanics is good enough so I can understand what they are saying, they can get full credit. The only grammatical advice I ever give is, "Never start your first sentence with a pronoun." Many of my students are as anxious about grammar and punctuation as they are about mathematics and statistics. For better or worse, I try to handle things so they never notice the writing in the course. My exams are what they are to reflect what statistics is about, not to reflect what writing is about.

However, there are some things on the border line between statistics and rhetoric that I do take into account. I prefer short, direct answers. (Often students are amazed at how short an answer I will accept.) Ambiguity or vagueness is taken as a sign of uncertainty and costs points. So do irrelevancies. I insist that students read the question carefully and stick to it. Indeed, the biggest problem I find (other than lack of knowledge of statistics) is failure to answer the question asked. This, of course, is a problem of thought rather than syntax.

I have been writing as if all the writing I require is on exams. That is very nearly true. Remember that I am trying to find ways to get students to think and ways to hold them accountable for thinking, and exams are the ultimate accountant. I have experimented with projects where students analyze a set of data and write up a report, but I have had less success with this. Just worrying about what the numbers mean is a wrenching change for many students. Asking them to consider the meaning of dozens of numbers and integrate them into a report is really too much to ask at the beginning. Perhaps this will change as other instructors, especially those in the high schools and grade schools, start to emphasize meaning and interpretation.

Another facilitator of change would be a better selection of textbooks. I mentioned earlier the reciprocal relation between reading and writing. Students are more likely to write well if their own textbook does so. Of course, the text should also support your emphasis on understanding and interpretation, while not making unrealistic assumptions about your students algebraic skills or reading level. I find Siegel's introductory statistics text to be excellent from this point of view.

In an age of assessment and accountability, perhaps I should close with some sort of "evaluation" of the success of what I have been doing. This is impossible. I have no idea of what students thought a standard deviation meant before I started asking them. Based on their answers during the brief transition period, before they expected such questions on exams, my suspicion is that it never dawned on them that a standard deviation *had* a meaning. It was just a cue-word used to Pavlovically stimulate a certain computation. On the other hand, I have often noticed that mathematicians and statisticians are among those *least* compelled to quantify everything, perhaps precisely because they *do* know the meanings of numbers—which entails knowing which numbers are meaningless. For me it is enough that today much of my students' attention is directed toward the parts of statistics that I consider most worth knowing. A decade ago almost all their attention was devoted to the parts least worth knowing. I cannot quantify that change, but I can tell you it is a very important change, and a change that could only have been brought about by making students write.

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