The Pursuit of Higher-Order Thinking in the Mathematics Classroom: A Review

Paul L. Goethals

This review was completed and submitted in partial fulfillment of the Master Teacher Program, a 2-year faculty professional development program conducted by the Center for Faculty Excellence, United States Military Academy, West Point, NY, 2013.

Abstract

Perhaps one of the most difficult skills or attributes to cultivate in a classroom environment is the ability of our students to think beyond traditional learning measures. Many studies suggest that teachers do not fully understand how to test, analyze, or even assess higher-order learning concepts. A wide number of educators also define it differently which adds to the complexity of the topic. This review begins with a description of higher-order thinking studies associated with mathematics in the classroom. Then, an annotated bibliography, as well as additional resources, is provided in an effort to advance the knowledge of higher-order thinking in mathematical concepts.

1. Introduction

1.1. *An Encounter with Higher-Order Thinking*

In the spring of 2003, Colonel Joseph Myers, a professor from the Department of Mathematical Sciences, was presenting several mathematical findings in history to a forum of cadets. Many of the cadets attending had to prepare a review of the presentation for another class, so they were busily taking notes as he spoke. The note-taking stopped, however, at one point in the presentation when Professor Myers asked the question, “Have any of you observed the fourth dimension?” He was referring specifically to the fourth *spatial* dimension, in contrast to the *temporal* dimension (time). Rather than just draw the fourth dimension, he went about building it, just as one would have done before the dawn of the computer. “In order to obtain the first spatial dimension, we connect two objects of dimension zero (points). Then, to obtain the second spatial dimension, we connect two one-dimensional objects.” He continued in this manner, as in Figure 1, with the cadets looking in disbelief, until the two cubes had been joined to form the fourth spatial dimension. Although the final picture on the dry erase board incorporated different colors so that the cadets could see the connections, it was clearly confusing in appearance – it didn’t seem to appear logical with overlapping lines. After a moment, one of the students commented that one of the cubes could be made smaller (*forming what we know as the tesseract*), thus removing the need to have overlapping lines.

1.2. *Refining the Research Space*

First and foremost, in literature associated with educational improvement and assessment, a distinction must be made among several similar topics. A wide number of educators have examined the role or assessment of *critical thinking,* and even *creative thinking,* in the classroom. Some researchers have investigated the art of *problem solving* or *analysis* as a means for developing higher-order thinking skills. *Metacognition*, referred to as "thinking about thinking" is also a frequent topic of interest. According to the Center for Development and Learning (CDL), a nonprofit organization dedicated to increasing success in academics, *higher-order thinking* includes all of these as subtopics. The organization defines higher-order thinking as including "concept formation, concept connection, getting the big picture, visualization, problem solving, questioning, idea generation, analytical (critical) thinking, practical thinking and creative thinking" (CDL, 2013). Bloom's taxonomy, which encompasses learning objectives approved by colleges nationwide, includes application, analysis, synthesis, and evaluation in the framework of higher-order thinking.

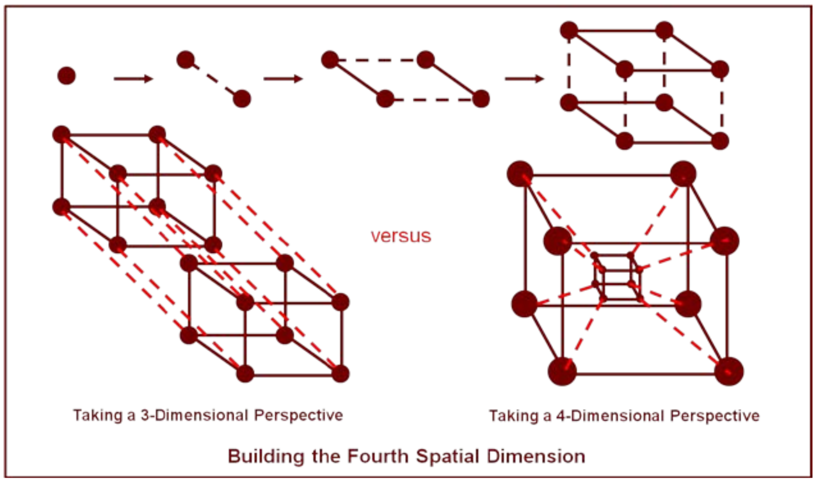


Figure 1: An Encounter with Higher-Order Thinking

Thus, the research in higher-order thinking by educators is vast – there are countless studies alone within the topic of critical thinking, considered a subfield to higher-order thinking. For this reason, this paper is scoped in two different ways: first, it is associated solely with the improvement or assessment of higher-order thinking in *mathematical studies*; and second, this paper will only examine the research that is subjectively deemed pertinent or supportive within the last *fifteen years*. These constraints are not meant to diminish or suggest that research prior to 1998 is not important – many of the research papers mentioned in this review are based upon previous findings or experiments performed prior to the 1990s. The motivation of this paper is only to cast light on the most contemporary research in higher-order thinking. A focus on mathematics is designed to further refine this broad topic area in alignment with the author's field of study.

1.3. *Research Foundation*

In the age of high performance computing where we attempt to analyze, decipher, and make inferences from immense volumes of data, the degree of complexity seems to require a deeper understanding of these problems. Therefore, it is easy to see the significance of research in higher-order thinking skills. The idea of pushing students to process material in new or innovative ways is not new in academia – for some time now, educators have been working to find specific techniques that relate to the effective analysis and synthesis of information. In many ways, however, our society continues to experiment with alternative teaching methods because it does not fully understand what works best or how some students are able to progress faster than others in mental aptitude. There are a wide number of educators who recognize the difficulty in testing, analyzing, or even assessing higher-order learning concepts.

And, while Bloom's taxonomy and other organizations *appear* to define higher-order thinking in a similar regard, there are a wide variety of definitions given by other researchers. Shown in Table 1 is just a brief account of the variation in meanings given to higher-order thinking over the past fifteen years.

Source	**Year**	**Definition**
King *et al*.	1998	"(It) includes critical, logical, reflective, metacognitive, and creative thinking. (It is) activated when individuals encounter unfamiliar problems, uncertainties, questions, or dilemmas."
NCTM	2000	"Solving a routine problem."
Anderson and Krathwohl	2001	The processes – analyze, evaluate, and create.
Lopez and Whittington	2001	"(It) occurs when a person takes new information and information stored in memory and interrelates and/or rearranges and extends this information to achieve a purpose or find possible answers in perplexing situations."
Weiss, E.	2003	Collaborative, authentic, ill-structured, and challenging problems.
Miri *et al*.	2007	"… the strategy – the setting of meta-objectives; whereas critical, systemic, and creative thinking are the tactics – the activities needed to achieve the proclaimed objectives."
Rajendran, N.	2008	The expanded use of the mind to meet new challenges.
Thompson, T.	2008	"Non-algorithmic thinking."
Thomas, A. and Thorne, G.	2010	"… (it) takes thinking to higher levels than just restating the facts. (It) requires that we do something with the facts. We must understand them, connect them to each other, categorize them, manipulate them, put them together in new or novel ways, and apply them as we seek new solutions to new problems."
Kruger, K.	2013	It involves "concept formation, critical thinking, creativity/brainstorming, problem solving, mental representation, rule use, reasoning, and logical thinking."

Table 1: Variation of Meanings – Higher-Order Thinking

For these reasons, it is anticipated that research in this topic will only continue to grow in depth and breadth. The National Council of Teachers of Mathematics (NCTM) and the National Research Council, a national advisory group in science and engineering, consider higher-order thinking to be an impetus for reforming future education in mathematics (NCTM, 2000; National Research Council, 2001). It is likely that standardized testing may also be affected, as a number of educators and administrators feel that it fails to assess problem solving or reasoning skills in students (Barksdale-Ladd and Thomas, 2000; Jones *et al*., 2003; Ravitch, 2010).

2. Higher-Order Thinking in Mathematics: A Review

In this section, a review of the literature associated with higher-order thinking in mathematics is presented. These research efforts can be categorized into two distinct areas – teaching techniques and assessment. Three examples of teaching techniques in mathematics are illustrated in the appendix.

2.1 *Teaching Techniques*

Many researchers have proposed using probing techniques or questioning methods to enhance thinking in students. The NCTM outlines a number of questions to ask math students depending on the grade level. Examples of those proposed by NCTM (2012) for helping students to conjecture, invent, and solve problems include:

(i) What would happen if …?
(ii) Do you see a pattern?
(iii) What are some possibilities here?
(iv) Can you predict the next one? How about the last one?
(v) How did you think about the problem?
(vi) What decision do you think he/she should make?
(vii) What is alike and what is different about your method of solution and his/hers?

Wetzel (2013) proposes using the following examples as questions that probe the individual toward higher understanding:

(i) What additional information do you need to solve the problem?
(ii) How does the data relate to your findings?
(iii) How does the evidence support your conclusions?
(iv) What would you need to do to determine if this solution is true?
(v) How can you compare this with other problems?

With questioning strategies, several researchers have looked solely at the role of this teaching technique in affecting mathematical output (William, 1999; Yee, 2000; Forster, 2004; Mok *et al*., 2008; Ratner and Epstein, 2009; Emerson, 2010).

Another teaching technique suggested by a number of researchers is to have students work in group settings. Collis (1998) and Paloff (1999) proposed working in groups for project-based graded events as a collaborative approach toward gaining knowledge. Springer *et al*. (1999) extended their study to examine cooperation in groups among a wide variety of graded events specific to science, mathematics, and engineering. A pilot research project performed by Mkhize (1999) examined cooperative work at the high school and college levels in mathematics. The project findings suggested that individuals enjoy working in groups more because they tend to gain a greater understanding of the material. This finding was not to suggest that the students did not trust their teacher, but rather, the students were more apt to ask probing questions and explore the problems more among their peers. Artzt *et al*. (2008) suggested that group work may be effective because students tend to challenge one another’s thinking abilities.

Perhaps one of the most frequent methods for promoting thinking in the classroom is centered upon using visualization techniques. For example, one may use software technology to demonstrate a number of different patterns, behaviors, or constructs (Figures 2-5).

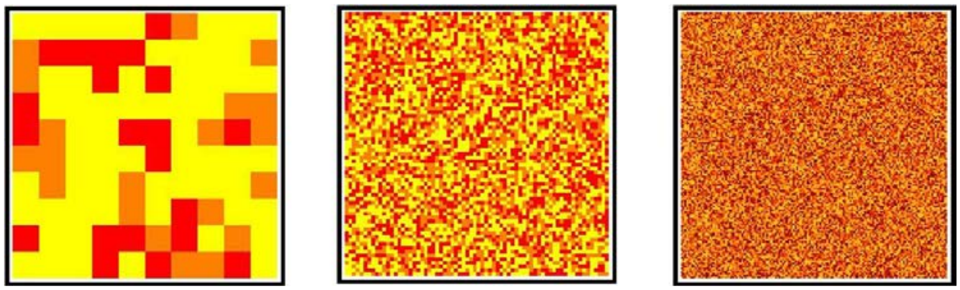


Figure 2: Visualizing Species Domination

Figure 3: Visualizing Territorial Competition, Matrix Composition, Trends or Forecasts

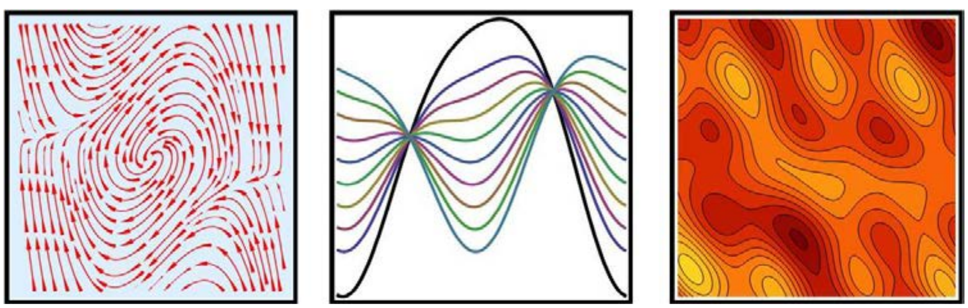


Figure 4: Visualizing Flow, Vibration, or Heat

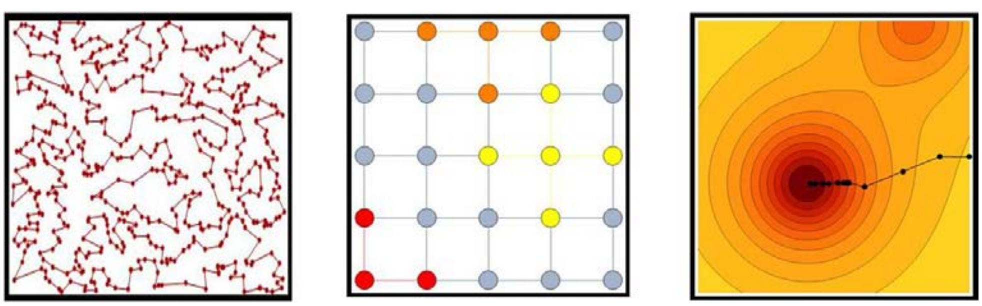


Figure 5: Visualizing Optimality in Terms of Paths, Networks, or Points

Although it is not necessary for better visualization, technology is frequently used in this respect. Rosen and Saloman (2007), O'Dwyer *et al*. (2008), Rosen (2011), and Weiss and Bordelon (2011) all sought to identify the effects of technology-rich environments on student performance in mathematics. Two researchers, Gianquinto (2007) and Rodd (2010) both suggested that there is some psychological attachment to imagery which causes one to receive an enhanced understanding of a concept. A wide number of researchers have performed case studies with specific mathematical images to compare and contrast student perception of their knowledge (Guzman, 2002; Presmeg, 2006; Souto and Gomez-Chacon, 2011). Recently, Gomez-Chacon (2013) reviewed a large number of visualization studies and proposed using interactive images as a modeling initiative to achieve higher-order thinking.

Furthermore, there are educators that sought specialized approaches or examined higher-order thinking strategies on a more broad level. Protheroe (2007) suggested that a mathematics classroom must do the following things to achieve an effective environment for higher-order thinking:

(i) Actively engage in doing mathematics
(ii) Solve challenging problems
(iii) Make interdisciplinary connections
(iv) Share mathematical ideas.
(v) Use multiple representations to communicate mathematical ideas
(vi) Use manipulatives and other tools.

In 2006, the Education Alliance (EA), a non-profit organization in West Virginia, collected a list of the most common successes from various math studies in conceptual understanding. The group proposed the following best teacher practices (EA, 2006):

(i) Focus lessons on specific concept/skills that are standards-based
(ii) Differentiate instruction through flexible grouping, individualizing lessons, compacting, using tiered assignments, and varying question levels
(iii) Ensure that instructional activities are learner-centered and emphasize inquiry/problem-solving
(iv) Use experience and prior knowledge as a basis for building new knowledge
(v) Use cooperative learning strategies and make real-life connections
(vi) Use scaffolding to make connections to concepts, procedures, and understanding
(vii) Ask probing questions which require students to justify their responses
(viii) Emphasize the development of basic computational skills

Upon studying the effects of varying instruction for high school math and science students, Miri *et al*. (2007) proposed the following three teaching strategies for generating higher-order thinking skills:

(i) Present real-world cases – encourage students to cope with relevant situations.
(ii) Direct class discussions related to a concept/phenomenon or a problem – encourage students to ask questions and present their own solutions.
(iii) Guide short inquiry-type experiments in groups – encourage students to learn in cooperation.

In addition, Staples and Truxaw (2010) sought to study the effects of altering the teaching discourse for math classes in an urban setting – emphasis was placed on the delivery or language used by the faculty in their study. They suggested that greater effort in education reform should be given to assessing the way that that the language of mathematics is administered in the classroom.

2.2. *Assessment*

For the difficulties in evaluating higher-order thinking in students, a wide number of researchers have used experimental case studies as a basis for studying assessment. Tests, grades, or the answers to specific questions are typically used as instruments in their studies. For instance, Newmann *et al*. (2001) used a statewide basic skills test given to third, sixth, and eighth grade students, to delineate between intellectual and non-intellectual questioning. They categorized the testing assessment process into two types – didactic, where students are evaluated based on their ability to recall definitions, know rules, or state facts, and interactive, in which problem solving and reasoning ability are required. A similar study was performed at the fourth-grade level by Wenglinsky (2004) – he examined the correlation between instruction emphasizing higher-order thinking and student performance on large-scale measures. Using the National Assessment of Educational Progress (NAEP) and the Trends in International Mathematics and Science Study (TIMSS) as his evidence, he suggested that there was a relationship between emphasis on thinking approaches and student performance.

Some research studies have found that faculty members are the primary reason for difficulties in assessing higher-order thinking. A recent study performed by Thompson (2008) examined the interpretation of Bloom's taxonomy by mathematics teachers in the southeastern U.S. The author found that math teachers do not fully understand the meaning of higher-order thinking and thus, have difficulty in creating test items for students.

Finally, there have been a number of researchers examining computer tools for assessing higher-order thinking in students. Thomas *et al*. (2002) incorporated a Java-based online assessment tool in their teaching of college-level statistics. Forgasz and Prince (2002) investigated a number of different software tools for assessing the understanding of mathematics. Lee *et al*. (2004) performed a similar study by integrating educational video games into second-grade mathematics classes to assess thinking skills. Rice (2007) suggests that the virtual appeal of computerize gaming environments can be used to improve cognitive skills. In addition, he identified several popular mathematics games in the computer market that failed to make this step. Others, such as Collis (1998), English and Yazdani (1999), Oliver and McLoughlin (2000), also promoted online virtual computer programs as tools for learning higher-level mathematical concepts.

References

1. Anderson, L., and Krathwohl, D. (eds.) (2001). *A Taxonomy for Learning, Teaching, and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives*. Longman Publishing Co., New York, New York.

2. Artzt, A., Armour-Thomas, E., and Curcio, F. (2008) *Becoming a Reflective Mathematics Teacher*. Lawrence Erlbaum Associates, New York, New York.

3. Barksdale-Ladd, M., and Thomas, K. (2000) What's At Stake in High Stakes Testing? *Journal of Teacher Education*, Vol. 51, No. 5, pp. 384-397.

4. Center for Development and Learning (2013) URL: http://www.cdl.org/index.html.

5. Collis, B. (1998) WWW-Based Environments for Collaborative Group Work. *Education and Information Technologies*, Vol. 3, pp. 231-245.

6. Emerson, J. (2010) *Connections Made Through Higher Level Questioning*. Master's Thesis, University of Nebraska - Lincoln.

7. English, S., and Yazdani, M. (1999) Computer-Supported Cooperative Learning in a Virtual University. *Journal of Computer Assisted Learning*, Vol. 15, No. 2, pp. 2-13.

8. Forgasz, H., and Prince, N. (2002) Software Used in Mathematics Learning – Reporting on a Survey. *Vinculum*, Vol. 39, No. 1, pp. 18-19.

9. Ghasempour, Z., Kashefi, H., Bakar, M., and Miri, S. (2012) Higher-Order Thinking via Mathematical Problem Posing Tasks among Engineering Students. *ASEAN Journal of Engineering Education*, Vol. 1, No. 1, pp. 41-47.

10. Giaquinto, M. (2007) *Visual Thinking in Mathematics: An Epistemological Study*. Oxford University Press, Oxford.

11. Gomez-Chacon (2013) Prospective Teachers' Interactive Visualization and Affect in Mathematical Problem Solving. The Mathematics Enthusiast, Vol. 10, Nos. 1/2, pp. 61-86.

12. Guzman, M. de (2002) The Role of Visualization in the Teaching and Learning of Mathematical Analysis. *Proceedings of the 2nd International Conference on the Teaching of Mathematics*, University of Crete, Greece.

13. Jones, M., Jones, B., and Hargrove, T. (2003) *The Unintended Consequences of High Stakes Testing*. Rowman and Littlefield, Inc., Lanham, Maryland.

14. King, F.J., Goodson, L., and Rohani, F. (1998) Higher-Order Thinking Skills: Definitions, Strategies, and Assessment. URL: http://www.cala.fsu.edu/files/higher_order_thinking_skills.pdf.

15. Kruger, K. (2013) Higher-Order Thinking. Hidden Sparks, Inc. New York, New York.

16. Lee J., Luchini, K., Michael, B., Norris, C., and Soloway, E. (2004) More Than Just Fun and Games: Assessing the Value of Educational Videos in the Classroom. *Proceedings of the Conference on Human Factors in Computing Systems*, pp. 1375 - 1378.

17. Lopez, J., and Whittington, M. (2001) Higher-Order Thinking in a College Course: A Case Study. *Proceedings of the Annual NACTA Conference*, Lincoln, Nebraska, pp. 22-29.

18. Miri, B., David, B.-C., and Uri, Z. (2007) Purposely Teaching for the Promotion of Higher-Order Thinking Skills: A Case of Critical Thinking. *Research in Science Education*, Vol. 37, pp. 353-369.

19. Mok, I., Kai, J., and Fung, A. (2008) Missing Learning Opportunities in Classroom Instruction: Evidence from an Analysis of a Well-Structured Lesson on Comparing Fractions. The Mathematics Educator, Vol. 11, Nos. 1/2, pp. 111-126.

20. National Council of Teachers of Mathematics (2000) *Principles and Standards for School Mathematics*. NCTM, Reston, Virginia.

21. National Research Council (2001) *Adding It Up: Helping Children Learn Mathematics*. Kilpatrick, J., Swafford, J., and Findell, B. (eds.), National Academy Press, Washington, D.C.

22. Newmann, F., Bryk, A., and Nagaoka, J. (2001) *Authentic Intellectual Work and Standardized Tests: Conflict or Coexistence?* Consortium on Chicago School Research, Chicago, Illinois.

23. O'Dwyer, L., Russell, M., Bebell, D., and Seeley, K. (2008) Examining the Relationship Between Student's Mathematics Scores and Computer Use at Home and School. *Journal of Technology, Learning, and Assessment*, Vol. 6, No. 5, pp. 4-45.

24. Oliver, R., and McLoughlin, C. (2000) Web-Based Learning and Generic Skills Development, *UniServe Science News*, Vol. 15, No. 1, pp. 3-7.

25. Paloff, R., and Pratt, K. (1999) *Promoting Collaborative Learning in Building Learning Communities in Cyberspace*. Jossey Bass, San Francisco, California.

26. Presmeg, N.C. (2006) Research on Visualization in Learning and Teaching Mathematics. *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future. PME 1976-2006*. ed. Sense Publishers, pp. 205–235.

27. Protheroe, N. (2007) What Does Good Math Instruction Look Like? *Principal*, Vol. 7, No. 1, pp. 51-54.

28. Rajendran, N. (2008) *Teaching and Acquiring Higher-Order Thinking Skills: Theory and Practice*. Tanjong Malim Perak: Penerbit Universiti Pendidikan Sultan Idris.

29. Ratner, A., and Epstein, S. (2009) *Reading Math Students: Math Educators Understanding Their Students as Readers*. Presentation at the Inaugural CUNY Education Conference, New York, New York.

30. Ravitch, D. (2010) *The Death and Life of the Great American School System: How Testing and Choice Are Undermining Education*. Basic Books, Philadelphia, Pennsylvania.

31. Rice, J. (2007) Assessing Higher Order Thinking inVideo Games. *Journal of Technology and Teacher Education*, Vol. 15, No. 1, pp. 87-100.

32. Rodd, M. (2010). Geometrical Visualization-Epistemic and Emotional. *For the Learning of Mathematics*, Vol. 30, No. 3, pp. 29-35.

33. Rosen, Y. (2011) The Effect of Teaching and Learning with Time To Know Environment on Math and English Language Arts Learning Achievements. *Proceedings of the Chais Conference on Instructional Technologies Research*, Raanana, Israel.

34. Rosen, Y., and Salomon, G. (2007) The Differential Learning Achievements of Constructivist Technology-Intensive Learning Environments as Compared with Traditional Ones: A Meta-Analysis. *Journal of Educational Computing Research*, Vol. 36, No. 1, pp. 1-14.

35. Shimazoe, J., and Aldrich, H. (2010) Group Work Can Be Gratifying: Understanding and Overcoming Resistance to Cooperative Learning. *College Teaching*, Vol. 58, pp. 52-57.

36. Souto, B., and Gomez-Chacon, I. (2011) Visualization at University Level: The Concept of Integral. *Annales de Didactique et de Sciences Cognitives*, Vol. 16, pp. 217-246.

37. Springer, L., Stanne, M., and Donovan, S. (1999) Effects of Small-Group Learning on Undergraduates in Science, Mathematics, Engineering, and Technology: A Meta-Analysis. *Review of Educational Research*, Vol. 69, No. 1, pp. 21-51.

38. Staples, M., and Truxaw, M. (2010) The Mathematics Discourse Learning Project: Fostering Higher Order Thinking and Academic Language in Urban Mathematics Classrooms, *Journal of Urban Mathematics Education*, Vol. 3, No. 1, pp. 27-56.

39. Thomas, A., and Thorne, G. (2010) *How to Increase Higher-Order Thinking*. Center for Development and Learning, Metairie, Louisiana.

40. Thomas, D., Okten, G., and Buis, P. (2002) On-line Assessment of Higher-Order Thinking Skills: A Java-Based Extension to Closed-Form Testing. Proceedings of the Sixth International Conference on Teaching Statistics, Cape Town, South Africa.

41. Thompson, T. (2008) Mathematics Teachers' Interpretation of Higher-Order Thinking in Bloom's Taxonomy. International Electronic Journal of Mathematics Education, Vol. 3, No. 2, pp. 96-109.

42. Vygotsky, L. (2002) *Thought and Language*, 13th ed. MIT Press, Cambridge, Massachusetts.

43. Weiss, D., and Bordelon, B. (2010) *The Instructional Design of Time To Know's Teaching Environment*. Graduate School of Education, Harvard University, Massachusetts.

44. Weiss, E. (2003) Problem-Based Learning in the Information Age: Designing Problems to Promote Higher-order Thinking. *Wiley Periodicals*, Vol. 95, pp. 25-31.

45. Wenglinsky, H. (2004) Facts of Critical Thinking Skills? What NAEP Results Say. *Educational Leadership*, Vol. 62, No. 1, p. 32.

46. William, Dylan (1999) Formative Assessment in Mathematics, Part I: Rich Questioning. *Equals: Mathematics and Special Educational Needs*, Vol. 5, No. 2, pp. 15-18.

47. Yee, F. (2000) Open-Ended Problems for Higher-Order Thinking in Mathematics. *Teaching and Learning*, Vol. 20, No. 2, pp. 49-57.

Annotated References

1. Brookhart, S. (2010) *How to Assess Higher-Order Thinking in the Classroom*. ASCD, Alexandria, Virginia.

I did not include this book in my review, as it was not specifically oriented to mathematics. Nonetheless, this is an excellent book outlining the assessment process for higher-order thinking skills. Brookhart provides us with the following outline:

Basic Assessment Principles:

(i) Begin by specifying clearly and exactly the kind of thinking, about what content, you wish to see evidence for.

(ii) Design performance tasks or test items that require students to use the targeted thinking and content knowledge.

(iii) Decide what you will take as evidence that the student has, in fact, exhibited this kind of thinking about the appropriate content.

Principles for Assessing Higher-Order Thinking

(i) Use introductory material.
(ii) Use novel material.
(iii) Manage cognitive complexity and difficulty separately.

2. Gartmann, S., and Freiberg, M. (1995) Metacognition and Mathematical Problem Solving: Helping Students to Ask the Right Questions. *The Mathematics Educator*, Vol. 6, No. 1, pp. 9-13.

The authors presented the results of a study examining several questioning strategies for higher-order thinking. They suggested that teachers should incorporate reflective discussion as a means to monitor thinking processes.

3. Forster, M. (2004) Higher-Order Thinking Skills. Australian Council for Educational Research, *Reynolds: Research Developments*, Issue 11, pp. 12-17.

This paper reports on the development of various programs in Australia for promoting higher-order thinking skills.

4. Miri, B., David, B.-C., and Uri, Z. (2007) Purposely Teaching for the Promotion of Higher-Order Thinking Skills: A Case of Critical Thinking. *Research in Science Education*, Vol. 37, pp. 353-369.

The authors presented the results of engaging high-school students with higher-order thinking skill problems in science. They suggested that if teachers specifically practice with real-world problems, encourage open-ended class discussion, and foster inquiry-oriented experiments, there is a greater likelihood that critical thinking capabilities will develop in students.

5. Muir, T., and Beswick, K. (2005) Where Did I Go Wrong? Students' Success at Various Stages of the Problem-Solving Process. *Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia*, Melbourne, Australia, pp. 561-568.

The authors presented the results of a large study regarding the influence of metacognitive mathematical thinking in 6th grade students. They suggested that the execution and verification stages of the problem-solving process have the greatest influence on successful higher-order thinking.

6. Wilson, J. (1998) Metacognition within Mathematics: A New and Practical Multi-Method Approach. *Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia*, Sydney, Australia, pp. 693-670.

This paper presents results associated with a Ph.D. pilot study, where students identify their actions when solving different types of mathematics problems. The author is investigating and reporting specifically on techniques for assessing and monitoring metacognition.

Several good links or online references:

1. University of Texas (2006). Algebra Readiness Toolkit. Retrieved from: http://www.ipsi.utexas.edu/docs/alg_readiness_toolkit/Admin%20white%20paper_NL_2-1-10.pdf

This outline presented by the University of Texas offers a very good comparison of skills-based and concepts-based instruction. A slight modification of the comparison between non-effective and effective instruction is presented below:

Ineffective Instruction	Effective Instruction
Students are shown step-by-step how to solve problems and the teacher expects them to do the problems exactly the way he/she does	The teacher asks the student to explain how he/she arrived at the answer to the problem.
The teacher ensures that his/her students do not get lost by requiring them to stop when they finish an assignment and wait for others to finish.	The teacher stimulates students' curiosity and encourages them to investigate further by asking them questions that begin with "What would happen if …?"
To keep them interested in math, the teacher works problems for his/her students and "magically" comes up with answers.	The teacher shows his/her students how "cool" math is and assures them that they all can learn it.
Two students are working problems on the board while the rest of the class watches.	The students in the class are talking to each other about math problems.
Students have been given 30 ordered pairs of numbers and are graphing them.	Students are working on creating a graph that shows the path of an approaching hurricane.
Students find the mean, median, and mode of a set of numbers.	Students are conducting an experiment, collecting the data and making predictions.
The students are sitting in rows and are quietly working on their assignment.	Students are sharing ideas while working in pairs or small groups.
At the end of class, the teacher collects everyone's worksheet and grades them.	Students have completed their work on chart paper and are holding the chart paper while explaining to the class how they reached their conclusions.
Students are in groups; one student in the group works out the problem while the others closely observe.	Students are acting out a problem in front of the class. Others in the class participate in a discussion of the problem.
The teacher is showing his/her students how they can use a formula to easily find the value of any term in a sequence.	Students are using color tiles to build the terms in a sequence.
The teacher believes that all students should get the same instruction at the same time. To accomplish this, he/she only uses whole group instruction.	Some students are working in groups, some in pairs and some individually. Not all students are working on exactly the same thing.

Table 2: Effective vs. Non-Effective Instruction in Mathematics

2. Yoong, W. K. (2002) Helping Your Students to Become Metacognitive in Mathematics: A Decade Later. Retrieved from: http://math.nie.edu.sg/kywong/Metacognition%20Wong.pdf.

3. King, F.J., Goodson, L., and Rohani, F. (1998) Higher-Order Thinking Skills: Definitions, Strategies, and Assessment. Retrieved from: http://www.cala.fsu.edu/files/higher_order_thinking_skills.pdf.

4. Wetzel, D. (commentaries):
http://www.teachscienceandmath.com/tag/higher-order-thinking-skills/
"How to Encourage Critical Thinking in Science and Math"
and http://www.teachscienceandmath.com/tag/best-practices-in-math-and-science/
"3 Best Practices of Successful Science and Math Teachers"

Useful Book:

Kulm, G. (ed.) (1990) *Assessing Higher Order Thinking in Mathematics*. American Association for the Advancement of Science, Washington, D.C.

A wide number of researchers cited this book; I did not include it in this review (outside the fifteen year constrain), but clearly it has had an impact on this literature topic.

Appendix – Classroom Examples

Example 1: Dot or Cross Product Lesson

1. If $\vec{\mathbf{a}} = \langle -2, 3, 8 \rangle$ and $\vec{\mathbf{b}} = \langle 4, 0, 5 \rangle$

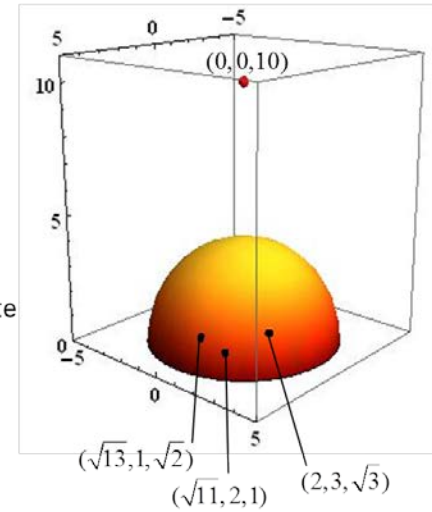
   a. Find $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$.

   b. Determine the magnitude of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

   c. Find the angle between the vectors.

versus

1. In the three-dimensional cartesian coordinate system shown, suppose that light rays are emitted from a source positioned at (0, 0, 10). Determine if the points indicated below on a spherical object actually receive light. Assume that all light rays emitted maintain a linear course (i.e. they are not bent in any way).



Example 2: Optimization Lesson

2. Given the function $f(x, y) = -2x^2 + 3xy$,

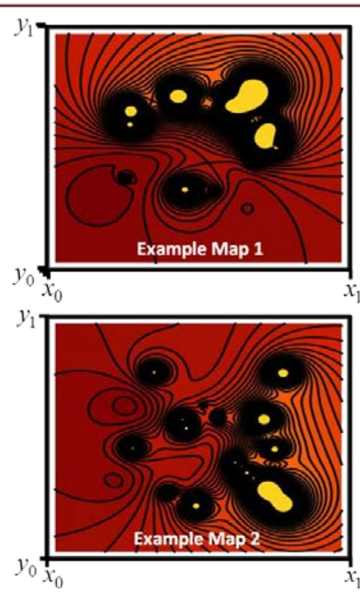
   a. Find the directional derivative at (1, 0).

   b. Determine the gradient of f, $\vec{\nabla} f$, at (2, 2).

   c. Find the maximum for the function.

versus

2. Given a function which describes the elevation of the terrain at any point (diagram), in a specified domain, $\{x_0 \le x \le x_1\}$ and $\{y_0 \le y \le y_1\}$, called the Area of Operations (AO), the U.S. Army would like you to develop a mathematical model for finding the optimal location for an observation post.



Example 3 – Relating Vector and Statistical Distribution Concepts (Real World Application)