Problem of the Week #12

The minimum number of papers remaining on the morning of day n is 6, since this number is $6/7$ of the total number of papers remaining after the previous morning. Working backwards, we find that $n=6$ is a possible solution, giving 36 papers originally.

We now show that this minimal solution is the only solution. We begin with a lemma.

Lemma: If $n \geq 2$ and $r \neq 0$, then $r^{n-1}+2r^{n-2}+3r^{n-3}+\cdots+(n-1)r = \dfrac{r^{n+1}-r+nr(1-r)}{(r-1)^2}$.

Proof: Let $S = r^{n-1}+2r^{n-2}+3r^{n-3}+\cdots+(n-1)r$. Multiplying both sides by r gives $rS = r^n+2r^{n-1}+3r^{n-2}+\cdots+(n-1)r^2$. Subtracting the first equation from the second gives $S(r-1) = r^n+r^{n-1}+r^{n-2}+\cdots+r^2-(n-1)r$. The first $n-1$ terms on the right side form a geometric series, with sum $r^n+r^{n-1}+\cdots+r^2 = r^2\left(\dfrac{r^{n-1}-1}{r-1}\right)$. Substituting and doing a little algebra gives the desired result.

Assume that there were x projects originally. There are $\frac{6}{7}(x-1)$ projects to grade at the end of the first day, $\frac{6}{7}\left(\frac{6}{7}(x-1)-2\right) = \left(\frac{6}{7}\right)^2 x - \left(\left(\frac{6}{7}\right)^2 + 2\left(\frac{6}{7}\right)\right)$ projects to grade at the end of the second day, $\frac{6}{7}\left(\left(\frac{6}{7}\right)^2 x - \left(\left(\frac{6}{7}\right)^2 + 2\left(\frac{6}{7}\right)\right) - 3\right) = \left(\frac{6}{7}\right)^3 x - \left(\left(\frac{6}{7}\right)^3 + 2\left(\frac{6}{7}\right)^2 + 3\left(\frac{6}{7}\right)\right)$ projects to grade at the end of the third day, and in general $\left(\frac{6}{7}\right)^{n-1} x - \left(\left(\frac{6}{7}\right)^{n-1} + 2\left(\frac{6}{7}\right)^{n-2} + \cdots + (n-1)\left(\frac{6}{7}\right)\right)$ projects to grade at the end of day $n-1$. This last number must equal n (if there is a solution to the problem), since the grading is finished on the morning of day n.

Therefore we have $\left(\frac{6}{7}\right)^{n-1} x - \left(\left(\frac{6}{7}\right)^{n-1} + 2\left(\frac{6}{7}\right)^{n-2} + \cdots + (n-1)\left(\frac{6}{7}\right)\right) = n$. Using the lemma with $r = \frac{6}{7}$ on the sum in the parentheses gives $\left(\frac{6}{7}\right)^{n-1} x - \dfrac{\left(\frac{6}{7}\right)^{n+1} - \left(\frac{6}{7}\right) + n \cdot \frac{6}{7}\left(1-\frac{6}{7}\right)}{\left(\frac{6}{7}-1\right)^2} = n$. After some tedious but straightforward algebra this equation can be massaged into the form $6^{n-1}(x-36) = 7^n(n-6)$.

Our original solution is now obvious: $x=36$ projects and $n=6$ days. (This actually guarantees only that the grading is finished on day n. We have to check that the number of projects at every step of the way is a positive integer, which is in fact the case.) It is easy to check that there are no solutions for $2 \leq n \leq 5$.

Consider the equation $6^{n-1}(x-36) = 7^n(n-6)$ and now assume that $n>6$. The integer $6^{n-1}(x-36)$ is divisible by 6^{n-1}, and thus $n-6$ on the right side must be divisible by 6^{n-1}. But an easy induction argument shows that $6^{n-1} > n-6$ for all $n>6$, so it is impossible for 6^{n-1} to be a factor of $n-6$. Thus there are no other solutions to the problem.

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