POTW #11-24 Circular Selection

Solving The Classic Problem Of Flavius Josephus

John Snyder, FSA
April 7, 2011
18:30 EDT

Problem

MAJ Lee Evans finally gets his “Discrete Dynamical Systems” club approved by the Directorate of Cadet Activities and decides to hold “tryouts” for the coveted position of club president. On the appointed day, x cadets arrive, each aspiring to be the DDS club president. MAJ Evans (who enjoys playing these strange games with cadets) has all those who want to be president stand in a circular formation. He walks up to the first cadet and dismisses him. Then moving counterclockwise (as is his preference), MAJ Evans orders the next cadet to “stand fast!” He then dismisses the third cadet unceremoniously, just as he did with the first, and orders the fourth to “stand fast!”

And so MAJ Evans makes his way counterclockwise, dismissing every other cadet that he comes to, around and around the circle until finally there is only one cadet remaining. This cadet is chosen to be the club president.

If you know the position of the first cadet (i.e., where MAJ Evans will start his selection process), where should you stand in the circle (as a function of x) in order to be the club president?

Solution

This is a variation of the classic Flavius Josephus problem which is discussed in detail on *Mathworld* and elsewhere. Using the *Combinatorica* package in *Mathematica* the problem can be solved by assigning a starting number of 0 to the first cadet who is immediately eliminated. Other cadets are then numbered 1, 2 ..., $x - 1$ counterclockwise around the circle. The cadet who will be chosen as president (i.e., the last cadet) is given as follows.

First we load the *Combinatorica* package.

```
In[1]:= Quiet@Needs["Combinatorica`"]
```

If there are originally x cadets then the number of positions counted counterclockwise around the circle from the first cadet to the cadet who is ultimately selected as president would be as follows for values of $x = 2, 3 ..., 26$ cadets.

```
In[2]:= test1 = Table[Last[System`InversePermutation[Josephus[n - 1, 2]]], {n, 2, 26}]

Out[2]= {1, 1, 3, 1, 3, 5, 7, 1, 3, 5, 7, 9, 11, 13, 15, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19}
```

For example, if there were originally $x = 7$ cadets then to be selected as president we should stand 5 positions around the circle from the first cadet measured in a counterclockwise direction.

A function of the original number of cadets x which gives these answers is:

$$position\ from\ first\ cadet = 1 + 2\,(x - 1) - 2^{\lfloor \log_2(x-1) \rfloor + 1}$$

Note the use of the floor function $\lfloor x \rfloor$ and the base 2 logarithm. Writing this as a function we test it against the above result.

```
In[3]:= president[x_] := 1 + 2 (x - 1) - 2^(Floor[Log[2,x-1]]+1);
```

In[4]:=
```
test2 = Table[president[x], {x, 2, 26}]
```

Out[4]=
```
{1, 1, 3, 1, 3, 5, 7, 1, 3, 5, 7, 9, 11, 13, 15, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19}
```

In[5]:=
```
test1 == test2
```

Out[5]=
```
True
```

And we see that it works. So if there are originally $x = 7$ cadets we should stand 5 places around the circle counterclockwise from the first cadet if we want to be selected president.

In[6]:=
```
president[7]
```

Out[6]=
```
5
```