# POTW #11-22  Cube Counting

## The Probability Of Correctly Reassembling A Painted Cube

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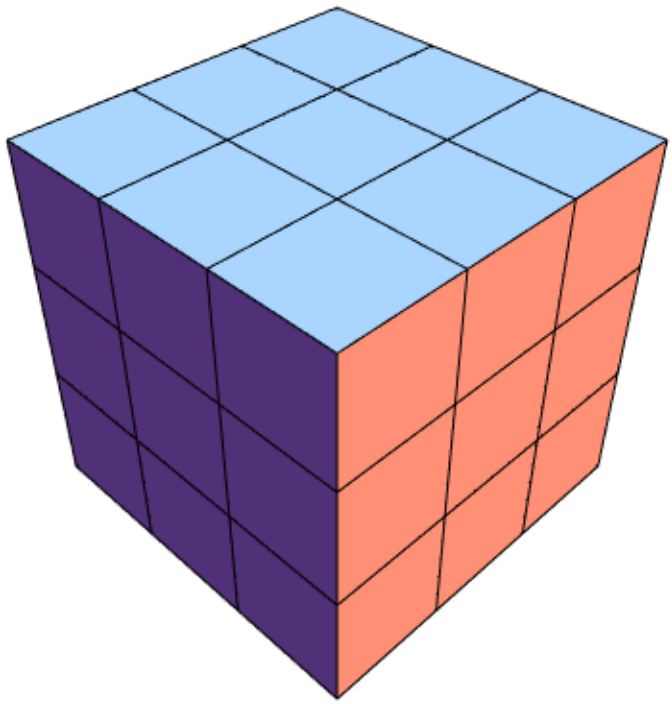
## Problem

Twenty-seven identical cubes are arranged into a larger 3x3x3 cube, and the outside of this larger cube is then painted black. Once the paint is dry the cube is disassembled, the 27 smaller cubes mixed, and then arbitrarily reassembled into a 3x3x3 cube. What is the probability that the entire outside of this newly formed cube is painted black?

## Solution

We start with a visualization of the sliced cube.



There are 8 corner pieces which are painted on 3 sides each. Consider the upper corner small cube nearest the reader's viewpoint in the drawing. Each cube has 6 faces any one of which could be placed facing outward along the light pink side of the cube. Once this face is selected the faces perpendicular to it can be rotated around it so that anyone of 4 faces appears on the light blue top of the cube. This makes $6 \times 4 = 24$ possible arrangements for each small cube. Of these only 3 arrangements of the corner cubes would be oriented to so that all of the black painted faces are pointing outward.

The chance of selecting a small cube painted on 3 sides for the first corner is 8/27 and the chance that it is then oriented so that all 3 painted sides face outward is 3/24. The number of ways of selecting and assembling the 8 corner cubes correctly is then:

In[1]:= $$\text{sides3} = \left(\frac{8}{27}\,\frac{3}{24}\right)\left(\frac{7}{26}\,\frac{3}{24}\right)\left(\frac{6}{25}\,\frac{3}{24}\right)\left(\frac{5}{24}\,\frac{3}{24}\right)\left(\frac{4}{23}\,\frac{3}{24}\right)\left(\frac{3}{22}\,\frac{3}{24}\right)\left(\frac{2}{21}\,\frac{3}{24}\right)\left(\frac{1}{20}\,\frac{3}{24}\right)$$

Out[1]= $$\frac{1}{37\,246\,677\,811\,200}$$

Next consider the 12 small cubes painted on 2 adjacent sides that fit in the center along the top, bottom and side edges of the large cube. Of the 24 possible orientations of these cubes only 2 will result in both the black sides facing outward as required. The chance of selecting these cubes and orienting them properly is then:

In[2]:= $$\text{sides2} = \left(\frac{12}{19}\,\frac{2}{24}\right)\left(\frac{11}{18}\,\frac{2}{24}\right)\left(\frac{10}{17}\,\frac{2}{24}\right)\left(\frac{9}{16}\,\frac{2}{24}\right)\left(\frac{8}{15}\,\frac{2}{24}\right)\left(\frac{7}{14}\,\frac{2}{24}\right)\left(\frac{6}{13}\,\frac{2}{24}\right)\left(\frac{5}{12}\,\frac{2}{24}\right)\left(\frac{4}{11}\,\frac{2}{24}\right)\left(\frac{3}{10}\,\frac{2}{24}\right)\left(\frac{2}{9}\,\frac{2}{24}\right)\left(\frac{1}{8}\,\frac{2}{24}\right)$$

Out[2]= $$\frac{1}{449\,264\,469\,386\,723\,328}$$

Now there are 6 small cubes painted on only 1 side that go in the center of the top, sides and bottom of the large cube. Of the 24 orientations of these cubes 4 will result in the painted side facing outward as required. The chance of selecting these cubes and orienting them correctly is then:

In[3]:= $$\text{sides1} = \left(\frac{6}{7}\,\frac{4}{24}\right)\left(\frac{5}{6}\,\frac{4}{24}\right)\left(\frac{4}{5}\,\frac{4}{24}\right)\left(\frac{3}{4}\,\frac{4}{24}\right)\left(\frac{2}{3}\,\frac{4}{24}\right)\left(\frac{1}{2}\,\frac{4}{24}\right)$$

Out[3]= $$\frac{1}{326\,592}$$

There is but a single unpainted small cube which goes in the center of the large cube. Because it is unpainted it can be oriented in any way. If all the other 26 small cubes are selected correctly this central cube must be properly positioned.

So the chance of assembling the cube in a random fashion so that the entire outside is painted black is:

In[4]:= $$\text{sides3} \times \text{sides2} \times \text{sides1}$$

Out[4]= $$\frac{1}{5\,465\,062\,811\,999\,459\,151\,238\,583\,897\,240\,371\,200}$$

This being about.

In[5]:= N[%]

Out[5]= $1.82981 \times 10^{-37}$

This solution can also be expressed in factorial notation as:

In[6]:= $$\frac{8! \times 12! \times 6!}{27!}\left(\frac{3}{24}\right)^{8}\left(\frac{2}{24}\right)^{12}\left(\frac{4}{24}\right)^{6}$$

Out[6]= $$\frac{1}{5\,465\,062\,811\,999\,459\,151\,238\,583\,897\,240\,371\,200}$$