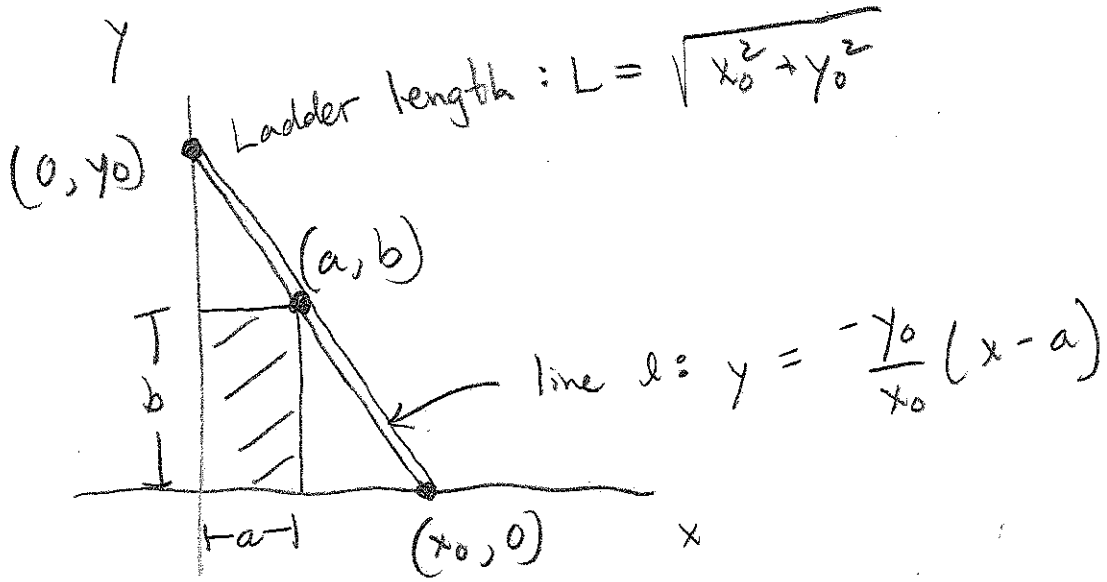
POTW : The ladder problem



(Note: $x_0 > a$ and $y_0 > b$)

Let $(x_0, 0)$ and $(0, y_0)$ be the coordinates where the ladder touches the floor and the wall resp.

Since (a, b) lies on ℓ, we have

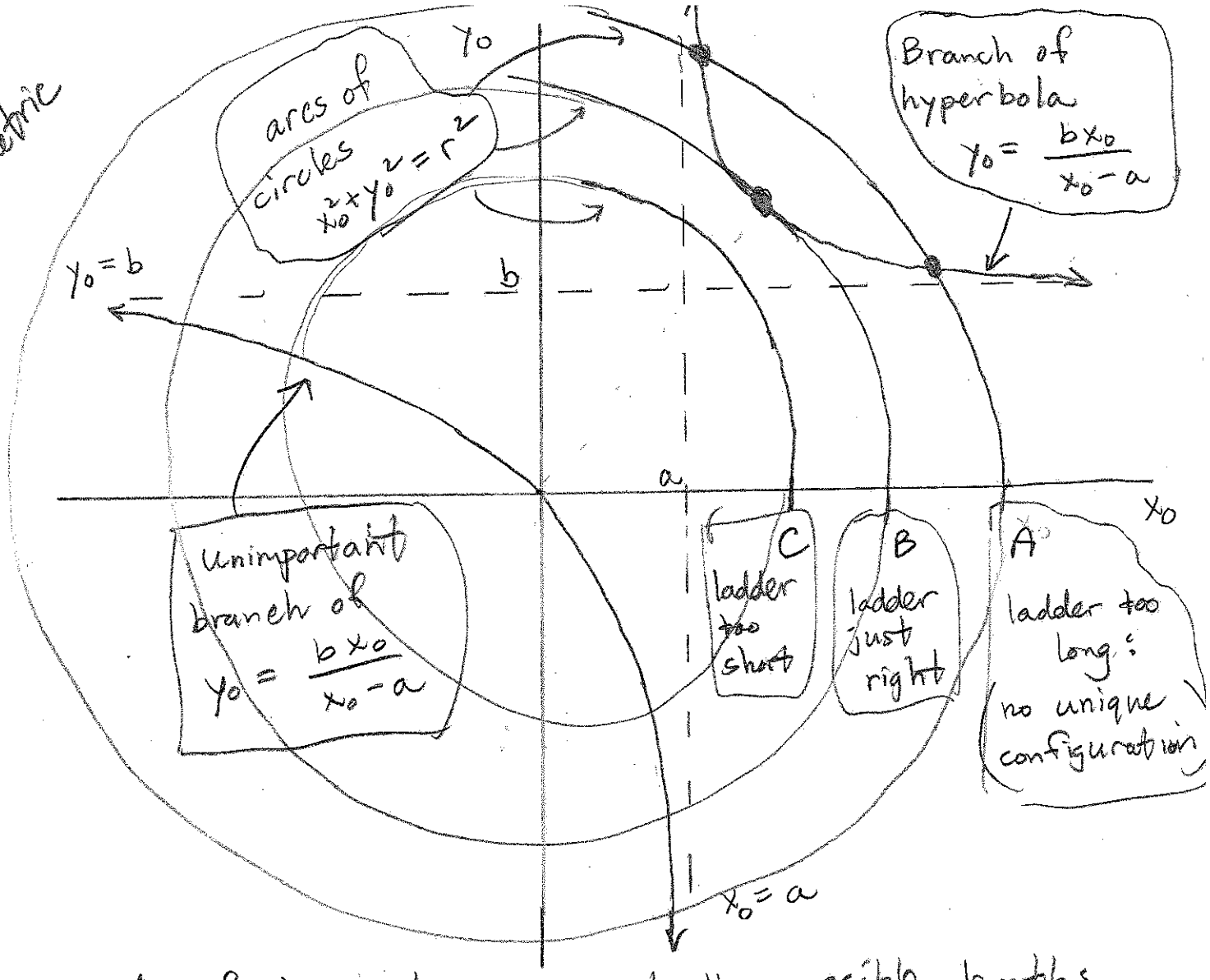
$$b = \frac{-y_0}{x_0}(a - x_0).$$

Solving for y_0, we obtain the constraint

$$(*) \qquad y_0 = \frac{b x_0}{x_0 - a}.$$

(2)

Geometric



The radii of the circles represent the possible lengths of ladders since $L^2 = x_0^2 + y_0^2$.

Ladder A is too long, ladder C is too short (it can't reach both the wall and the floor <u>and</u> touch the box).

Ladder B is the correct length since it touches the graph of the constraint (*) exactly once.

③ Analytical

The radius of the circle represented by ladder B is the solution, L.

L is equal to the minimum distance between the branch of the hyperbola $y_0 = \frac{b x_0}{x_0 - a}$, $x_0 > a$ and the origin in the $x_0 y_0$-plane.

$$L = \min_{x_0 > a} d\left((0,0), \left(x_0, \frac{b x_0}{x_0 - a}\right)\right)$$

$$= \min_{x_0 > a} \sqrt{x_0^2 + \left(\frac{b x_0}{x_0 - a}\right)^2}$$

Define $s(x_0) := x_0^2 + \left(\frac{b x_0}{x_0 - a}\right)^2$ (dist. squared)

$$\left(\frac{ds}{dx_0} = 0 \text{ and } x_0 > a\right) \Rightarrow x_0 = a + a^{1/3} b^{2/3}$$ (clearly corresponds to a min)

Call this value $\bar{x}$, $\bar{x} := a + a^{1/3} b^{2/3}$

Therefore the ladder's length is

$$L = \sqrt{s(\bar{x})} = \sqrt{\left(a + a^{1/3} b^{2/3}\right)^2 + \frac{\left(a + a^{1/3} b^{2/3}\right)^2 b^{2/3}}{a^{2/3}}}$$

When $a = 27$ and $b = 64$, we get that

$$L = \sqrt{(27 + 3 \cdot 16)^2 + \frac{(27 + 3 \cdot 16)^2 \cdot 16}{9}} = \underline{\underline{125 \text{ units}}}$$