Dog problem of the week

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Let

$$\begin{aligned} \vec{F}(t) &= (F_1(t), F_2(t)), \\ \vec{L}(t) &= (L_1(t), L_2(t)), \\ \vec{R}(t) &= (R_1(t), R_2(t)), \\ \vec{C}(t) &= (C_1(t), C_2(t)), \end{aligned}$$

denote the xy-position of the four dogs, whose names are Fido, Lassie, Rin Tin Tin, and Cujo respectively.

WLOG assume Fido starts at $(0, 0)$, Lassie starts at $(a, 0)$, RTT starts at (a, a) and Cujo starts at $(0, a)$. Also assume that each dog feels an irresistible urge to move towards the dog counter-clockwise from him (which is obviously his left!).

Then Fido moves in the direction of the vector $\vec{L}(t) - \vec{F}(t)$, Lassie moves in the direction $\vec{R}(t) - \vec{L}(t)$, RTT moves in the direction $\vec{C}(t) - \vec{R}(t)$, and Cujo moves in the direction $\vec{F}(t) - \vec{C}(t)$. Thus the dynamics can be modelled by the 8-dimensional linear IVP

$$\begin{pmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{L}_1 \\ \dot{L}_2 \\ \dot{R}_1 \\ \dot{R}_2 \\ \dot{C}_1 \\ \dot{C}_2 \end{pmatrix} = \begin{pmatrix} L_1 - F_1 \\ L_2 - F_2 \\ R_1 - L_1 \\ R_2 - L_2 \\ C_1 - R_1 \\ C_2 - R_2 \\ F_1 - C_1 \\ F_2 - C_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ L_1 \\ L_2 \\ R_1 \\ R_2 \\ C_1 \\ C_2 \end{pmatrix}, \tag{1}$$

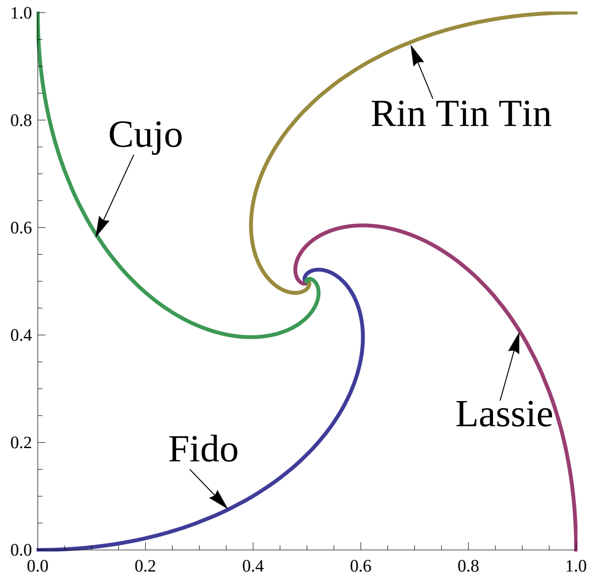

Figure 1: Path of each dog with $a = 1$.

with initial condition

$$\begin{pmatrix} F_1(0) \\ F_2(0) \\ L_1(0) \\ L_2(0) \\ R_1(0) \\ R_2(0) \\ C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ a \\ a \\ 0 \\ a \end{pmatrix},$$

where the dots in equation (1) denote time derivatives.

Using the eigenvalues and eigenvectors of the coefficient matrix we get the solution to (1),

$$\begin{pmatrix} F_1(t) \\ F_2(t) \\ L_1(t) \\ L_2(t) \\ R_1(t) \\ R_2(t) \\ C_1(t) \\ C_2(t) \end{pmatrix} = \frac{a}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{a}{2} e^{-t} \begin{pmatrix} \sin t - \cos t \\ -\sin t - \cos t \\ \sin t + \cos t \\ \sin t - \cos t \\ -\sin t + \cos t \\ \sin t + \cos t \\ -\sin t - \cos t \\ -\sin t + \cos t \end{pmatrix}. \tag{2}$$

See Figure 1 above for the plot of each dog's path. By symmetry, each dog travels the same distance as it approaches the equilibrium value at $(a/2, a/2)$. That distance is

$$\int_0^\infty \left| \frac{d\vec{F}}{dt} \right| dt = \int_0^\infty \left| \frac{d\vec{L}}{dt} \right| dt = \int_0^\infty \left| \frac{d\vec{R}}{dt} \right| dt = \int_0^\infty \left| \frac{d\vec{C}}{dt} \right| dt = a$$