POTW #12-19 Maximum Cone Volume

Finding The Volume Subject To A Surface Area Constraint

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Problem

An open-top cone has a surface area of 1 m^2. What is the *maximum possible volume* that the cone can hold?

Solution

In the majority of what follows we assume that the cone is a right circular cone.

- **Solve As A Single Variable Problem**

We can solve the problem as a single variable optimization problem. If the radius of the base is r and the height is h the volume of the cone is $\frac{1}{3}\pi r^2 h$ and the surface area (excluding the base) is $\pi r \sqrt{r^2 + h^2}$. We solve for the height h in terms of the base radius r using the fact that the surface area must be 1 m^2.

In[1]:=
```
Solve[π r Sqrt[r^2 + h^2] == 1 && h > 0 && r > 0, h, Reals] // PowerExpand
```

Out[1]= $\left\{\left\{h \to \text{ConditionalExpression}\left[\frac{\sqrt{1-\pi^2 r^4}}{\pi r}, 0 < r < \frac{1}{\sqrt{\pi}}\right]\right\}\right\}$

Plugging this into the volume formula we have:

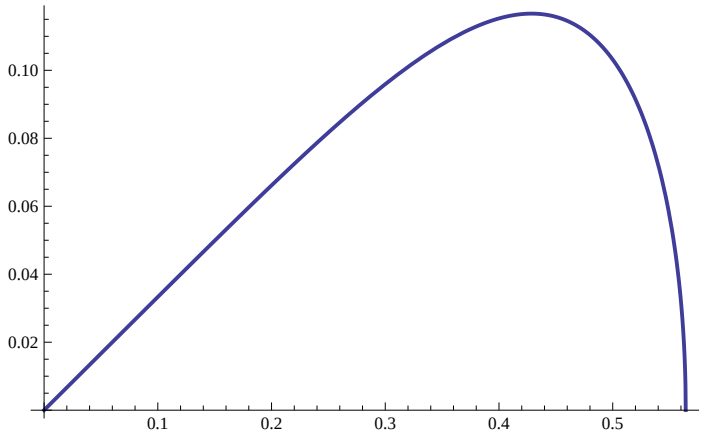
In[2]:=
```
1/3 π r^2 h /. % // FullSimplify[#, 0 < r < 1/Sqrt[π]] & // First
```

Out[2]= $\frac{1}{3} r \sqrt{1-\pi^2 r^4}$

We can plot this function.

In[3]:=
```
Plot[1/3 r Sqrt[1 - π^2 r^4], {r, 0, 1/Sqrt[π]}, PlotStyle → Thick]
```

Out[3]=



There is clearly a maximum volume when r is around 0.43. We can find the maximum volume by taking the derivative, setting it equal to zero, and solving for r.

In[4]:=
```
Solve[D[1/3 r Sqrt[1 - π^2 r^4], r] == 0 && 0 < r < 1/Sqrt[π], r] // First
```

Out[4]= $\left\{r \to \frac{1}{3^{1/4}\sqrt{\pi}}\right\}$

And plugging this value of r into the volume formula we find the maximum volume.

In[5]:=
```
1/3 r Sqrt[1 - π^2 r^4] /. % // Simplify
```

Out[5]= $\frac{\sqrt{\frac{2}{\pi}}}{3 \times 3^{3/4}}$

In[6]:=
```
% // N
```

Out[6]= 0.116675

Solution Using Lagrange Multipliers

If we let λ be a Lagrange multiplier we can solve for the maximum volume as follows.

In[7]:=
```
Λ = 1/3 π r^2 h + λ (π r Sqrt[r^2 + h^2] - 1);
```

Taking the derivative with respect to r, h, λ and then setting them equal to zero we obtain three equations.

In[8]:=
```
eqs = Thread[D[Λ, #] & /@ {r, h, λ} == 0] // Simplify
```

Out[8]= $\left\{\frac{2 h \pi r}{3} + \frac{\pi \left(h^2 + 2 r^2\right) \lambda}{\sqrt{h^2 + r^2}} == 0,\ r\left(r + \frac{3 h \lambda}{\sqrt{h^2 + r^2}}\right) == 0,\ \pi r \sqrt{h^2 + r^2} == 1\right\}$

Solving these equations over the real numbers we have.

In[9]:=
```
sol = Solve[eqs, {r, h, λ}, Reals]
```

Out[9]=
$$\left\{\left\{r \to \frac{1}{3^{1/4}\sqrt{\pi}},\ h \to -\frac{\sqrt{\frac{2}{\pi}}}{3^{1/4}},\ \lambda \to \frac{1}{3^{3/4}\sqrt{2\pi}}\right\},\ \left\{r \to \frac{1}{3^{1/4}\sqrt{\pi}},\ h \to \frac{\sqrt{\frac{2}{\pi}}}{3^{1/4}},\ \lambda \to -\frac{1}{3^{3/4}\sqrt{2\pi}}\right\}\right\}$$

Taking the positive roots we obtain the maximum volume of:

In[10]:=
```
vol = 1/3 π r^2 h /. sol[[2]]
```

Out[10]=
$$\frac{\sqrt{\frac{2}{\pi}}}{3 \times 3^{3/4}}$$

In[11]:=
```
% // N
```

Out[11]= 0.116675

Solution Using MaxValue

With *Mathematica* the problem is more easily solved using the `MaxValue` function.

In[12]:=
```
MaxValue[{1/3 π r^2 h, π r Sqrt[r^2 + h^2] == 1}, {r, h}]
```

Out[12]=
$$\frac{\sqrt{\frac{2}{\pi}}}{3 \times 3^{3/4}}$$

In[13]:=
```
% // N
```

Out[13]= 0.116675

Solution Assuming An Elliptical Cone

It should be pointed out that we can achieve a larger volume subject to the same surface area constraint if we are willing to consider cones which are not right circular cones. In the case of an elliptic cone, for example, the volume is $\frac{1}{3}\pi a b h$ and the surface area is $2a\sqrt{b^2+h^2}\ E\left(\sqrt{\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{h^2}}}\right)$, where $E(k)$ is a complete elliptic integral of the second kind. Using these formulas we can numerically find a cone of larger volume as we see in the next cell.

In[14]:=
```
NMaximize[{1/3 π a b h, 2 a Sqrt[b^2 + h^2] EllipticE[Sqrt[(1 - b^2/a^2)/(1 + b^2/h^2)]] == 1, a ≥ 0, b ≥ 0, h ≥ 0}, {a, b, h}]
```

Out[14]= {0.130893, {a → 0.49787, b → 0.394993, h → 0.635599}}

This cone is found to contain about 12.2% more volume than the corresponding right circular cone.

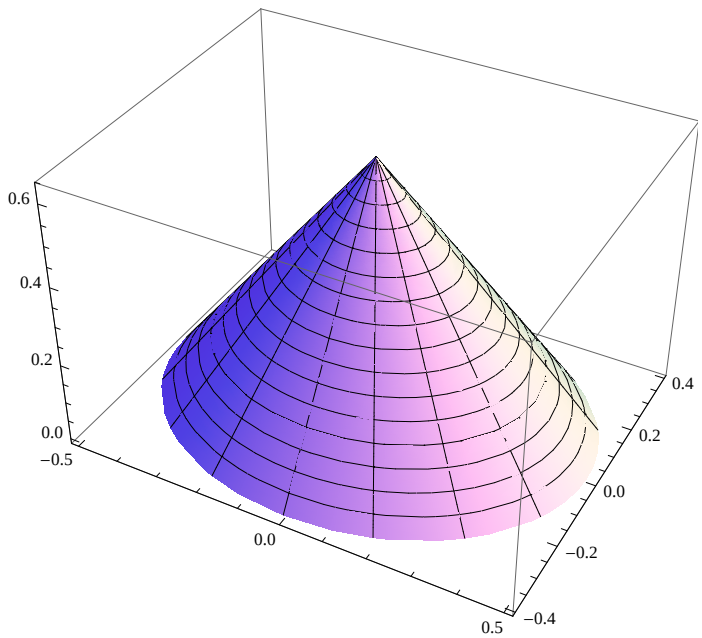
In[15]:=
```
0.130893 / 0.116675
```

Out[15]= 1.12186

This cone looks like the following visualization having an elliptical base.

In[16]:=
```
Block[{a = 0.49787, b = 0.394993, h = 0.635599},
 ParametricPlot3D[{a (h - u)/h Cos[v], b (h - u)/h Sin[v], u},
  {v, 0, 2 π}, {u, 0, h}, BoxRatios → Automatic]]
```

Out[16]=



We might be able to find a cone of even larger volume having a surface area of 1 m^2 by using the calculus of variations. That, however, is a problem for another day.