POTW #21

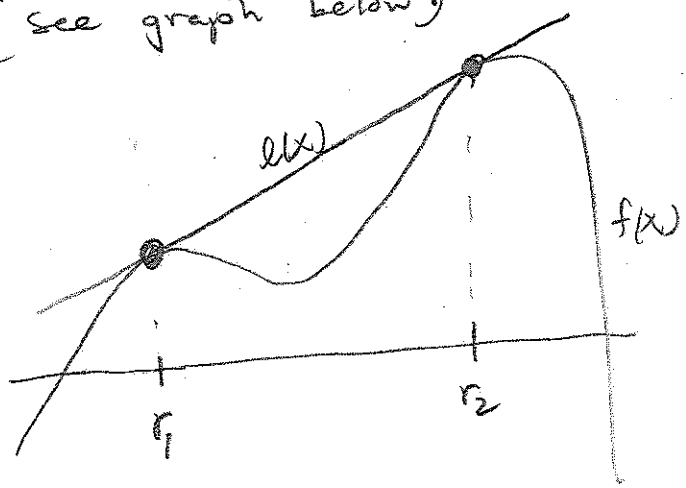
Let $l(x) = mx + b$ be a line tangent to two points $r_1, r_2 \in [1,7]$ on the graph of

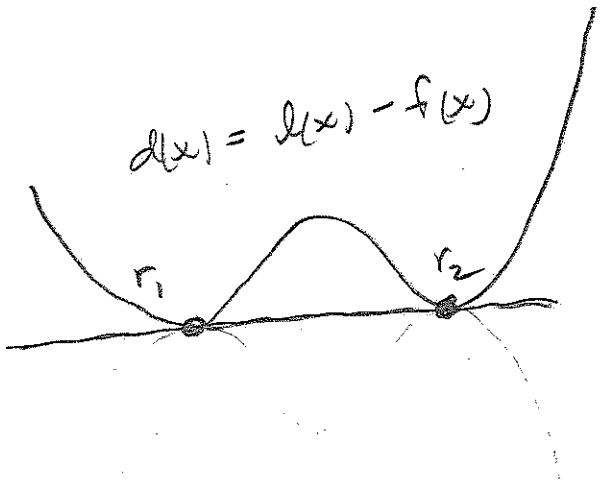
$$f(x) = -x^4 + 15x^3 - 76x^2 + 155x - 100.$$

Then the function

$d(x) = l(x) - f(x)$ is a polynomial of degree 4 with repeated roots at r_1 and r_2.

(see graph below)





Thus
$$\begin{aligned} d(x) &= (x-r_1)^2(x-r_2)^2 \\ &= x^4 + \left(-2(r_1+r_2)\right)x^3 + \left(r_1^2 + 4r_1r_2 + r_2^2\right)x^2 \\ &\quad + \left(-2r_1r_2(r_2+r_1)\right)x + r_1^2 + r_2^2. \end{aligned}$$

Equating the coefficients of d leads to the system of eqns:

$$-2(r_1+r_2) = -15$$
$$r_1^2 + 4r_1r_2 + r_2^2 = 76$$
$$-2r_1r_2^2 - 2r_1^2r_2 = m - 155$$
$$r_1^2r_2^2 = 100 + b$$

which has a solution w/

$$m = 55/8, \quad b = -\frac{159}{64}$$

Thus

$$\boxed{l(x) = \frac{55}{8}x - \frac{159}{64}}$$

- Woaster