PROBLEM OF THE WEEK #18 (RESTROOM CALCULUS)

Consider a roll of single-ply restroom toilet tissue. The entire roll is 12 inches in diameter. The unusable portion (including the hole in the middle and the thickness of the cardboard tube) is four inches in diameter. If the toilet tissue is .01 inches in thickness, and each sheet is five inches in length, how many sheets of toilet tissue, rounded to the nearest whole number, are on the roll? *(Assume that there is no air space between the layers of paper.)*

Solution proposed by Philippe Fondanaiche, Paris, France

Three methods are possible to estimate the number of sheets of toilet tissue

1) **Area method**
   We calculate the area of the annulus between the cardboard tube and the entire roll, that is to say $A = \pi.(6^2 - 4^2) = 32\pi$
   As the thickness T of the tissue is equal to .01 inches, the total length of the spiral is equal to $L = A/T = 3200\pi$
   As each sheet is 5 inches in length, the number of sheets of toilet tissue is equal to 640π rounded to **2011**

2) **Summation method (approximate method)**
   Let R = 2 inches the distance between the center of the roll and the first turn of the spiral et let T the thickness of the tissue. Considering that a turn of a spiral of angle 2π starting at the distance d of the center of the roll can be approximated by the circonference of a circle of radius d, we have the following equation providing the total length L of the tissue:
   $L=2\pi R + 2\pi(R + T) + 2\pi(R + 2T) + 2\pi(R + 3T) + ...+ 2\pi(R + [N -1]T)$ with N representing the total number of turns which is equal to $(6 – 2)/.01 = 400$
   So $L = 2\pi NR + \pi N(N-1)T) = \pi N(2R + (N-1)T) = 400\pi(4+3.99)=3196\pi$.
   The corresponding number of sheets is equal to **2008**.

3) **Calculus method based on the archimedean spiral**
   The archimedean spiral is defined by the equation in polar coordinates $r = A\theta$ with :
   a) the constant A such that $2\pi A = T$ which is the thickness of the tissue.
   b) θ is the angle in radians swept out as the spiral increases from the radius of 0 to the radius r.

   To reach the surface of the cardboard tube, the angle θ increases from 0 to θ_1 such that

   $r = 2$ inches $= A\theta_1$. So $\theta_1 = 2/A = 4\pi/T = 400\pi$.

To reach the surface of the roll, the angle θ increases from 0 to θ_2 such that $\theta_2 = 6/A = 1200\pi$.

The formula from calculus providing the total length of the spiral is the following:

$$L = \frac{1}{400\pi}\left[\theta\sqrt{1+\theta^2} + \mathrm{Ln}(\theta + \sqrt{1+\theta^2})\right]_{\theta_1}^{\theta_2}$$

See the appendix for the detailed calculation of this formula.

Finally, with the values of θ_1 and θ_2 above mentioned , we get L = 11309,74106 - 1256,64369 = 10053,0974..

So the number of sheets is equal to **2011**.

It appears that the results of the methods n°1 and n°3 are logically concordant.

Appendix

The formula from calculus for the arc length of a curve $r = h(\theta)$ in polar coordinates between $\theta = c$ radians and $\theta = d$ radians is:

$$L(c,d) = \int_c^d \sqrt{h'(\theta)^2 + h(\theta)^2}\,d\theta$$

For the Archimedean spiral $h(\theta) = \frac{T}{2\pi}\theta,\ h'(\theta) = \frac{T}{2\pi}$, so this arc length is given by

$$(2) \qquad L(c,d = \frac{T}{2\pi}\int_c^d \sqrt{1+\theta^2}\,d\theta$$

This is an interesting integral to calculate! We will carry out the integration later but here is the indefinite integral given in one table of integrals:

$$\int \sqrt{1+\theta^2}\,d\theta = \frac{1}{2}\left[\theta\sqrt{1+\theta^2} + \mathrm{Ln}(\sqrt{1+\theta^2} + \theta)\right] + c$$

We can express the solution (2) in terms of a function $f[\theta_d]$ of a single varible θ_d. That variable θ_d represents the angle in radians starting from 0 radians and wrapping counter clockwise around pole to the angle $\theta = \theta_d$ radians:

$$f(\theta_d) = \frac{T}{2\pi}\int_0^{\theta_d} \sqrt{1+\theta^2}\,d\theta = \frac{T}{4\pi}\left[\left[\theta\sqrt{1+\theta^2} + \mathrm{Ln}(\sqrt{1+\theta^2} + \theta)\right]\right]_0^{\theta_d}$$