PROBLEM OF THE WEEK #20

Country x is planning to attack country y, and country y is anticipating the attack. Country x can either attack by land or by sea. Country y can either prepare for a land defense or a sea defense. Both countries must choose either an all land or all sea strategy; they may not divide their forces. The following are the probabilities of a successful invasion according to all possible strategy combinations.

- If x launches an attack by sea and y prepares a defense by sea, the probability of a successful invasion is 80%.
- If x launches an attack by sea, and y prepares a defense by land the probability of a successful invasion is 100%.
- If x launches an attack by land and y prepares a defense by land, the probability of a successful invasion is 60%.
- If x launches an attack by land and y prepares a defense by sea, the probability of a successful invasion is 100%.

Assume that the goal of country x is to maximize the probability of a successful invasion.

Assume that the goal of country y is to minimize the probability of a successful invasion.

a) What is the optimal strategy for country x?

b) What is the final probability of a successful invasion assuming both countries utilize an optimal strategy?

Solution proposed by Philippe Fondanaiche

In a first step, we consider that the country x and the country y choose their strategy respectively according to the following probabilities:

1) x attacks y by sea with probability p and by land with probability 1 – p.
2) y defends by sea with probability q and by land with probability 1 – q .

So we have the table giving the probabilities of invasion according to the four combinations:

			x attacks by	
			sea	land
			p	1- p
y defends	sea	q	**0.8**	**1**
by	land	1 - q	**1**	**0.6**

The probability of a successful invasion is given by the relation:

$P(p,q) = 0.8pq + (1 - p)q + p(1 - q) + 0.6(1 - p)(1 - q) = 0.2(2p + 2q - 3pq + 3)$

As the goal of the country y is to minimize this probability, the derivative of P(pq,) with respect to q provides the equation $- 3p + 2 = 0$ or $p = 2/3$. So the country x should attack by sea with probability 2/3 and by land with probability 1/3 whereas with the same reasoning the country y should also defend by sea with probability $q = 2/3$ and by land with probability $1 - q = 1/3$.

Therefore the probability of successful invasion is $0.2\ (4/3 + 4/3 - 4/3 + 3) = 13/15 = 86{,}66..\%$