Mathematical Modeling in the USMA Curriculum

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Introduction

Mathematical modeling is an important and fundamental skill for quantitative problem solvers. Increasingly, Army officers are called upon to solve problems that can be modeled mathematically. It is therefore important that the United States Military Academy (USMA) curriculum includes mathematical modeling and problem solving. Many of the principles, skills, and attitudes necessary to perform effective mathematical modeling can be presented in the core mathematics courses taken by all cadets.

After providing some background about the USMA program, we briefly discuss the mathematical modeling process. We then give examples of course topics, developmental problems, and student projects that are used to accomplish early presentation of modeling and problem solving. This "early and often" approach to teaching modeling results in undergraduates who are able to apply their mathematics to solve quantitative problems. This paper also serves to demonstrate how modeling supports the innovative mathematics curriculum presented at USMA. We discuss and explain large-scale modeling projects called Interdisciplinary Lively Application Projects (ILAPs) in the context of the curriculum.

Background

"A mind is not a vessel to be filled, but a flame to be kindled" -- Plutarch

USMA was founded in 1802 as the nation's first engineering school and the first national educational institution. In 1817, newly appointed Academy Superintendent Sylvanus Thayer, previously an assistant professor of mathematics, returned to USMA from a visit to the Ecole Polytechnique in Paris and instituted a rigorous four-year engineering curriculum. He placed great emphasis on modeling and applied mathematics. The basis of civil engineering at the time was descriptive geometry, with algebra, trigonometry, and some calculus supporting the necessary mechanics of engineering. Thayer and his Professor of Mathematics, Charles Davies, converted the theoretical approach of the French at the Ecole Polytechnique to the applied modeling approach needed in America. Modeling became the thread holding together the rigorous and substantial undergraduate core mathematics program at West Point. In turn this applied mathematics and modeling approach was exported from USMA to other technical schools around the country during the 19th century.

By the middle of the 20th century less curriculum time was available for mathematics, yet more sophisticated mathematical topics were being required of cadets. The core curriculum dilemma of fitting seven topics (Differential Calculus, Integral Calculus, Multi-variable Calculus, Differential Equations, Linear Algebra, Probability and Statistics, Discrete Math) into the four allotted semesters of core mathematics was a substantial challenge for USMA and many other technically based schools.

West Point instituted a new integrated mathematics curriculum in 1990. The four semester courses; Discrete Dynamical Systems and Introduction to Calculus, Calculus 1 (with differential equations), Calculus 2 (multivariable), and Probability & Statistics, satisfied the “7 into 4” demands at West Point. Modeling was an important thread linking the four courses together for student growth and development. The integration of the four core courses made for a more sophisticated multiple perspective of modeling. Students investigate and traverse behavior, models, and solution methods using discrete or continuous, linear or nonlinear, and deterministic or stochastic mathematics. The following figure shows the possibilities of the modeling flow just considering discrete and continuous classifications. Similar flow perspectives are possible for linear--nonlinear and stochastic--deterministic classifications.

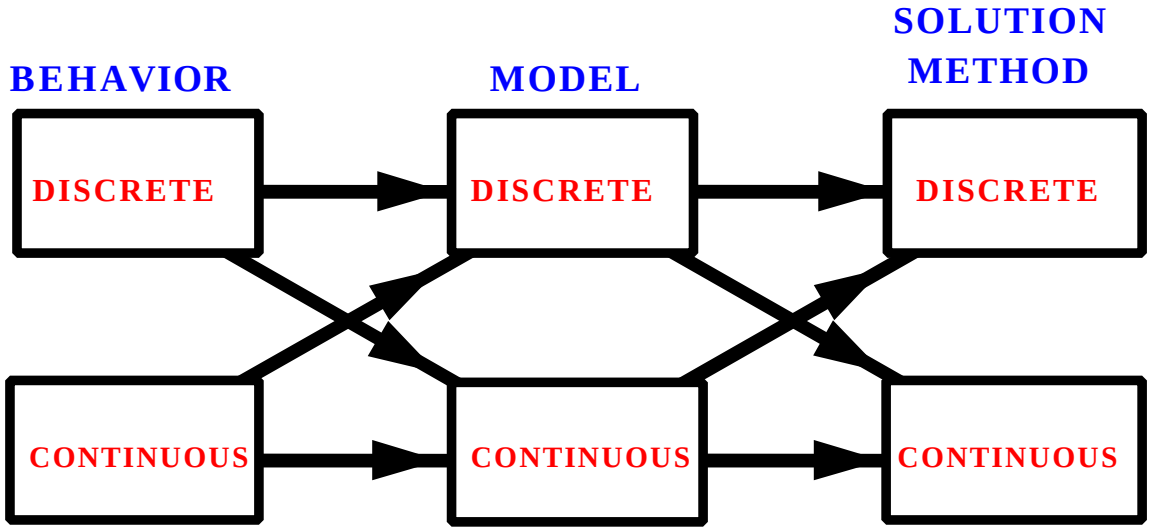

Figure 1: Modeling flow possibilities.

Another feature of the USMA mathematics core curriculum is the use of Interdisciplinary Lively Applications Projects (ILAPs). These projects are broader and, in some ways, more realistic than previous educational modeling projects used at West Point. ILAPs are co-authored and co-presented by mathematicians and experts from partner disciplines in the USMA program (for example, engineers, scientists, and economists). The disciplinary experts present background information to the students. Students then model and solve

the problem in small groups, write their solution and present their results. Finally, disciplinary experts critique the problem and explain the use of mathematics in the applied discipline. Students benefit by seeing a more realistic, multidisciplinary problem, and faculty benefit by working together to develop educational projects using modeling and writing as key elements in problem solving.

Integrated Curriculum

"The greatest good you can do (for students) is not just to share your riches, but to reveal (to them) their own..." -- Benjamin Disraeli

The education philosophy of the Department of Mathematical Sciences at West Point is that undergraduate students should acquire important and fundamental knowledge, develop logical thought processes, and learn how to learn. Successful students can formulate intelligent questions, reason and research solutions, and are confident and independent in their work. Modeling plays a vital role in supporting this philosophy.

With this philosophy in mind, the core mathematics curriculum integrates concepts of the seven fundamental engineering-based mathematics topics into the core program. A key to this core curriculum effort is a first course containing discrete dynamical systems (difference equations) including systems of equations (matrix/linear algebra) and a sequential approach to differential calculus. The major ingredients of this course are simple proportionality models and applications. The second course, Calculus I, covers integral calculus, differential equations, and calculus-based models of a single variable. Many of the applications and models covered in a discrete mode in the first course are revisited in a continuous mode in the second core course. By the end of the first year, students have the ability to understand the perspective of problem solving presented in Figure 1. The third course, Calculus II, includes topics in

multivariable and vector calculus. Students encounter more sophisticated models in higher dimensions with more complex geometries. The fourth core mathematics course at USMA is Probability & Statistics. In this course, students use stochastic modeling to revisit previous problems with new mathematical perspectives and new fundamental concepts, and to solve new application problems.

Throughout these four core courses, the major content themes of undergraduate mathematics are studied using new and different perspectives. These content themes are functions, limits, change, accumulation, vectors, approximation, visualization, representation of models, and solution methods. In addition to modeling and writing, a computation thread ties course content together. The role of technology in this program includes and extends beyond using tool for calculations, exploration and discovery. Technology aids students in visualization and graphics, data analysis, communication, and integration of various means of problem solving.

Mathematical Modeling

Mathematical modeling is the process of formulating and solving real-world quantitative problems using mathematics. In performing this process, we normally need to describe a real-world phenomenon or behavior in mathematical terms. Often, the problem solver is interested in understanding how a system works, the cause of its behavior, the sensitivity of the process to changes, predicting what will happen, or making a decision based on the mathematical model developed.

The four basic steps in the mathematical modeling process are as follows:

Step 1: Identify the Problem
Step 2: Develop a Mathematical Model
Step 3: Solve the Model
Step 4: Verify, Interpret, and Use the Model

We briefly discuss each of these steps which are not always so clear and distinct in every problem-solving endeavor. However, by using this mathematical modeling process, problem solvers can gain confidence to approach complex and difficult problems and even develop their own innovative approaches to solving problems.

Step 1: Identify the Problem

The problem needs to be stated in as precise a form as possible. One must understand and consider the scope of the problem when writing this statement. Sometimes, this is an easy step, while other times this may be the most difficult step of the entire modeling process.

Step 2: Develop the Mathematical Model

Developing the model entails both translating the natural language statement made in Step 1 to a mathematical language statement and understanding the relationships between factors involved in the problem. To further understand and define these relationships, simplifying assumptions are usually needed. In this step, the problem solver defines variables, establishes notation, and identifies some form of mathematical relationship and/or structure. The mathematical model is sometimes the equivalent of the problem statement in mathematical notation.

Step 3: Solve the Model

This step is usually the most familiar to students. The model from Step 2 is solved, and the answer understood in the context of the original problem. The problem solver may need to further simplify the model if it cannot be solved. The solution procedure many times involves analytic, numeric, and/or graphic techniques.

Step 4: Verify, Interpret, and Use the Model

Once solved, the model must be tested to verify that it answers the original problem statement, it makes sense, and it works properly. After verifying the model, the problem solver interprets its output in the context of the problem. It is possible that the model works fine, but the modeler could develop or needs to develop a better one. It is also possible that the model works, but it's too cumbersome or too expensive to use. Once again, the problem solver returns to earlier steps to adjust the model until it meets the desired and necessary criteria. Once the model meets the design needs, one uses the model to solve the problem.

As indicated, the modeling process is iterative in the sense that as the problem solver proceeds, he or she may need to go back to earlier steps and repeat the process or continue to cycle through the entire process or part of it several times. "Simplifying the model" refers to the process of going back to make the model simpler because it cannot be solved or is too cumbersome to use. "Refining the model" indicates the process of going back to make the model more powerful or to add more complication. By simplification and refinement, one can adjust the realism, accuracy, precision, and robustness needed for the model.

We should mention that the model is just a tool to solve the problem. How well the problem solver navigates the entire modeling process determines the success or failure of the problem-solving endeavor.

Early Modeling

Modeling in undergraduate core courses is often used to predict or explain simple changing behavior, such as proportionality or linear growth or decay. Even early modelers need to be exposed to the power and limitations of modeling. One of the most fundamental concepts discussed in early courses is when to use modeling to solve quantitative problems. Once the modeling thread is started, the emphases for beginning modelers are stating and understanding underlying assumptions and defining variables. As the modelers become more experienced, they explore and discuss the sensitivity of the conclusions to the assumptions, and more rigorously approach the verification stage of the modeling process.

The discrete dynamical systems course is ideal for introducing the fundamental concepts of modeling and problem solving. Proportionality behaviors often produce constant coefficient dynamical models. Further refinements can produce nonlinear equations or systems of equations, which are still solvable through iteration. Conjecturing models and solutions and verifying their accuracy are natural processes in this first course of dynamical systems.

As continuous modeling is introduced in the Calculus I course, differential equations are often produced as models. Recent developments in computing packages make analysis of differential equations and systems of equations accessible to undergraduate students. Computers and calculators are natural tools to help the student in several stages of the modeling process.

In both discrete and continuous dynamics, the basic concept is that the future is predicted by first understanding the present and then conjecturing the change over the interval of interest. These dynamical models are solvable numerically by iteration or by approximation and iteration. These topics provide accessibility for freshmen to the modeling process. The heat conduction problem given in the following example is the type of problem that many mathematics programs traditionally cover in higher-level courses (junior or senior year). Now, this type of problem is accessible to first year cadets taking the discrete dynamical modeling course at USMA. The mathematical modeling process illustrates how a freshman student using difference equations can determine the transfer of heat through a metal bar (or rod).

Example: Heat Conduction in a Metal Bar

Step 1: Identify the Problem:

Find a function representing the temperature along the length of the interior of a metal bar at different times given known temperatures at the ends of the bar.

Step 2: Develop a Mathematical Model:

The problem solver will use a discrete view of the bar by selecting evenly spaced points on the bar. To produce a basic model, we pick three evenly spaced points on the interior of the bar along with the two endpoints: See Figure 2 to visualize this geometry. The problem solver desires to describe the change of temperature at each of the points over an interval of time (a time step). In this model the problem solver assumes that the set of points and intervals of time accurately approximate the reality of continuous temperature change. "A" is the set temperature at Point 0 (left endpoint) and "B" is the set temperature at Point 4 (right endpoint).

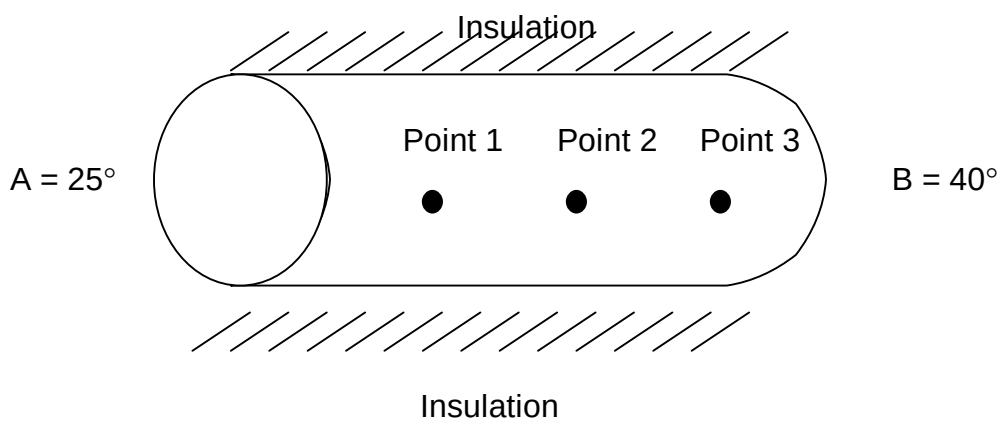

Figure 2. Geometry of the bar.

The problem solver defines $t_1(n)$, $t_2(n)$, and $t_3(n)$, to be the temperature at time step *n* at the points 1, 2, and 3 respectively. An assumption is that the surface temperature of the rod does not affect the internal temperatures (rod is insulated). The major effect on the temperature at a point is then the temperature at the points next to it. For this 3-point geometry, temperature at Point 1 affects that at Point 2, the temperature at Point 2 affects that at Points 1 and 3, and the temperature at Point 3 affects that at Point 2. If the temperature $t_1(n)$ is higher than $t_2(n)$ then Point 1 causes the temperature at Point 2 to increase. The amount of temperature increase, $t_2(n+1)-t_2(n)$, is proportional to the amount Point 1 is hotter than Point 2, $k(t_1(n)-t_2(n))$, where *k* is the proportionality constant. Similarly, if the factor $t_1(n)-t_2(n)$ is negative, Point 1 causes the temperature at Point 2 to decrease.

The difference model of the influence of the temperature at Point 1 on that at Point 2 is given by $t_2(n+1)-t_2(n)=k\left(t_1(n)-t_2(n)\right)$. Modeling continues by analyzing the influence of the temperature at Point 3 on that at Point 2. In a very similar manner, the difference equation model for this situation is $t_2(n+1)-t_2(n)=k\left(t_3(n)-t_2(n)\right)$. The total change in one time step of the temperature at Point 2 is the sum of these two effects. Therefore, the difference model for Point 2 is $t_2(n+1)-t_2(n)=k\left(t_1(n)-t_2(n)\right)+k\left(t_3(n)-t_2(n)\right)$. The temperature at Point 1 is influenced from the right by the temperature at Point 2 and from the left by the endpoint temperature *A*. Therefore, the function describing the change over time of the temperature at Point 1 is $t_1(n+1)-t_1(n)=k\left(t_2(n)-t_1(n)\right)+k\left(A-t_1(n)\right)$. Similarly, the temperature at Point 3 is influenced from the left by the temperature at Point 2 and the right by the endpoint temperature *B*. The change over time of the temperature at Point 3 is modeled by $t_3(n+1)-t_3(n)=k\left(t_2(n)-t_3(n)\right)+k\left(B-t_3(n)\right)$.

We assemble the entire system of difference equations that describe the change of temperature of the three points on the bar. The equations along with the values of the temperatures at an initial time give us a model for predicting the temperature at later times. The first order, linear, and homogeneous system of difference equations (the mathematical model) is given by:

$$
\begin{aligned}
t_1(n+1) &= (1-2k)t_1(n)+k\,t_2(n)+Ak \\
t_2(n+1) &= k\,t_1(n)+(1-2k)t_2(n)+k\,t_3(n) \\
t_3(n+1) &= k\,t_2(n)+(1-2k)t_3(n)+Bk
\end{aligned}
$$

Step 3: Solve the Model:

The problem solver can use iteration or analytic solution methods from linear algebra to solve this model. For this example, given specific values for *k, A, B,* and initial values for $t_1(0)$, $t_2(0)$, $t_3(0)$, we give only the graphic portrayals of the solution for three different times. With k=0.25, A=25, B=40, and $t_1(0)$=0, $t_2(0)$=0, $t_3(0)$=0, the plots for n=0, 1, 2, 3 are shown in Figure 3.

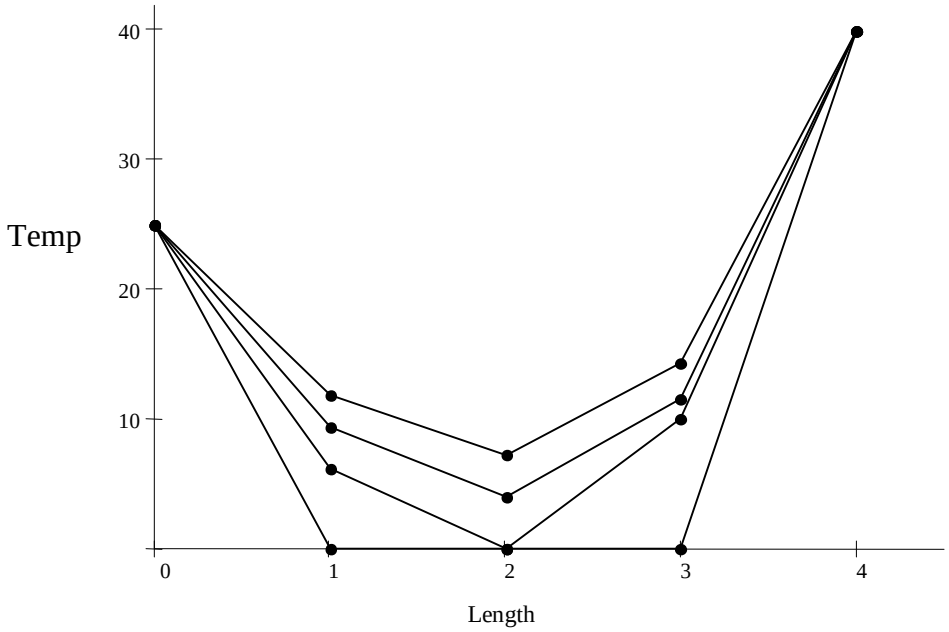

Figure 3. Temperature in the bar for t = 0, 1, 2, 3 (n = 0, 1, 2, 3).

Step 4: Verify, Interpret, and Use the Model:

The plots in Figure 3 show the temperature rising in the interior of the bar and moving towards a more uniform (and possibly linear) function. This makes sense given the assumptions. However, the problem solver may want to refine the model by adding more discretization points in the bar and allowing the model to proceed for more iterations. Graphing these results may provide a better visualization of the temperature change on the insulated bar. Although not presented in this example, the modeler often needs this type of refinement step.

Interdisciplinary Lively Application Projects (ILAPs)

"The heart of mathematics consists of concrete examples and concrete problems." -- Paul Halmos (1970)

At USMA and the other 15 schools involved in a consortium called Project INTERMATH, cooperative efforts by faculty from mathematics and partner departments produce application projects called ILAPs. These problems offer opportunities for students to apply mathematical modeling procedures. Groups of students solve these applied problems using modeling, computing, and reasoning. They present the results of their modeling work orally and in writing. Finally, the expert authors of the ILAPs present critiques and

extension briefings for the students. The ILAPs provide motivation for students by showing them the relevance of their mathematics. They also enable the consolidation of concepts in the curriculum, give students experiences in group and interdisciplinary problem solving, and vividly demonstrate the relevance of mathematical modeling skills.

Summary

Modeling can help set the standard for undergraduate mathematics education. In the past, mathematics was understood as the language of science. Today mathematics can also provide the structure for scientific reasoning and quantitative problem solving. In the mathematics course work at USMA, we emphasize the entire modeling process--developing, solving, interpreting, and communicating with the emphasis on analyzing realistic, interdisciplinary problems. Our mathematics students will be more valuable to the military as active and skilled members of problem solving teams. Through modeling in core courses, we can make progress in developing these more powerful quantitative problem solvers. These future Army officers will understand and demand proper quantitative reasoning. Modeling is both an appropriate and valuable thread in early mathematics courses at the United States Military Academy.

Exercises

1. Write an essay that explains mathematical modeling and its role in problem solving. Relate the modeling process to your own problem solving experiences.

2. Investigate the process called the “Scientific Method.” How is this process similar to and different from the mathematical modeling process?

3. List some quantitative problem solving principles you have learned or think may be helpful in your own problem solving. Could these principles apply to solving problems in a military setting?