The Modified Covering Problem on Paths and Trees

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OUTLINE

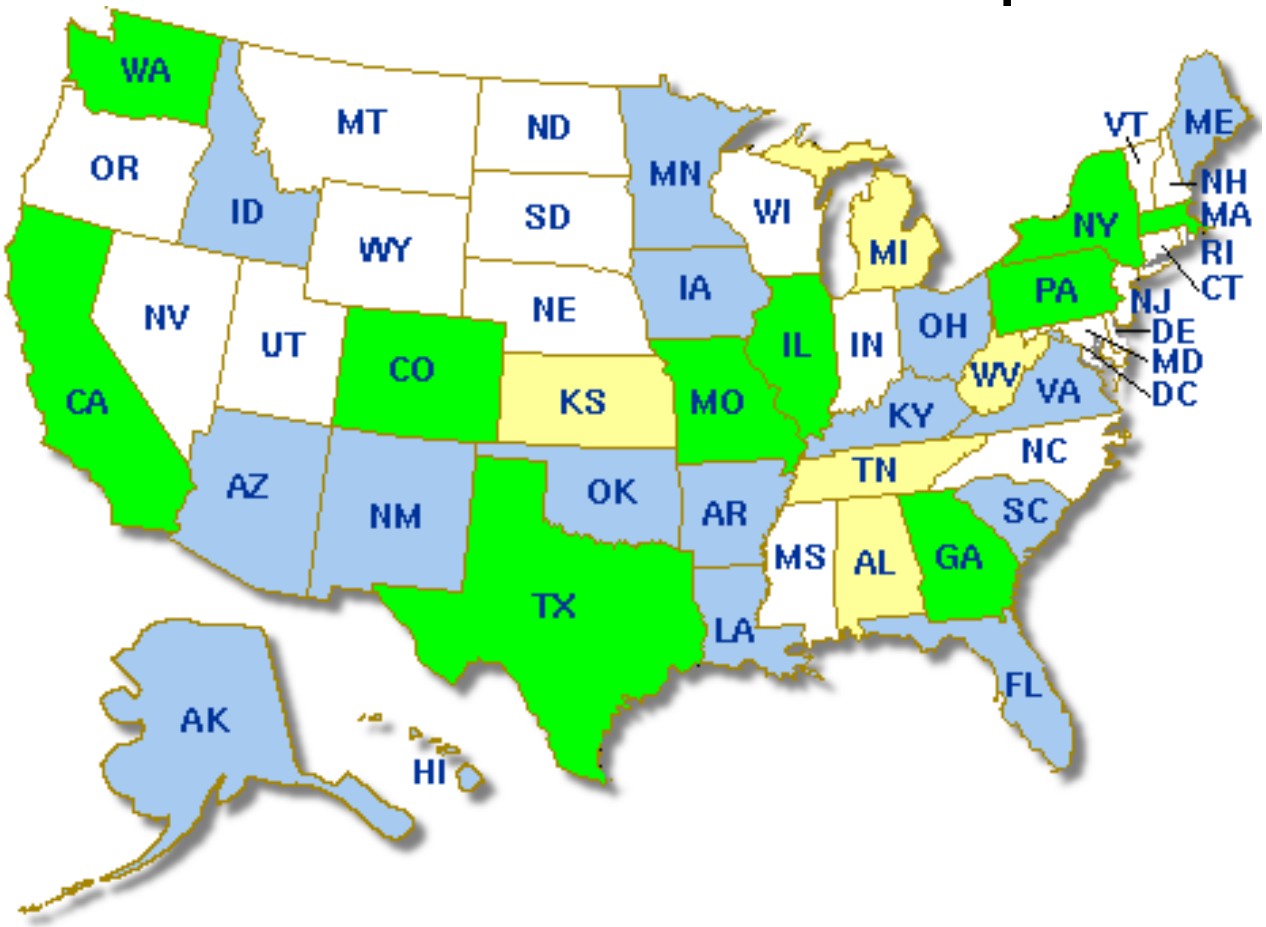
- Motivations – Civil Defense & Military Operations
- The Modified Covering Problem (MCP)
- Examining the MCP on Paths and Trees
  - Paths with uniform length edges
  - Paths with non-uniform length edges
  - Bounding the solution on trees
- Ongoing Research
- Future Research

THE MODIFIED COVERING PROBLEM

- Locate the minimum number (or cost) of facilities such that:
  - Each demand is within a specified distance (or time) of a facility
  - No demand is covered by a co-located facility.

MOTIVATION

- ARNG WMD Civil Response Teams



Authorizations:
FY1999 – 10
FY2000 – 17
FY2001 – 5

Criteria:
10 FEMA Districts
Population
Response time

MOTIVATION

- Coverage
  - “provides optimal response coverage for the majority of the United States population.”
- Response times
  - Albany WMD-CST reportedly took 12 hours to respond on-site to 11SEP 01 NYC incident
- Funding
  - $60M in FY1999, $75M in FY2000,
  - Every state wants their own
- Politics

OTHER MOTIVATIONS

- General network
  - Location of artillery units among existing base camps
- On paths
  - Location of reserve forces behind the FLOT along a linear front
  - Location of QRFs along an MSR

THE MCP – Definitions

- Set Covering Problem
- Modified Covering Problem
  - Interfacility covering
    - Unit demand incapacitates co-located facility
    - Facility cannot serve co-located demand
  - Identical facility capability
  - Set of available facility locations = set of demand locations

THE MCP - Notation

Decision Variables:

x_j = 1 if a facility is located at site j, 0 otherwise;

Sets and Set Notation:

N = {i,j: i,j=1, 2, …, n}, the index set of demands and available facility sites, $x_i = x_j$;

Other Variables:

c_j = fixed cost of opening a facility at site $j \in N$

b_{ij} =1 if a facility at site j can cover a demand at site i

0 otherwise ($b_{kk}=0 \ \forall \ k \in N$)

THE MCP - Formulation

minimize $z = \Sigma_j c_j x_j$

subject to $\Sigma_j b_{ij} x_j \geq 1 \quad \forall\, i \in N$

$x_j \in \{0,1\} \quad \forall\, j \in N$

<u>Note</u>: $b_{jj}=0 \;\forall\, j \in N$

MCP on PATHS & TREES

- Previous efforts with CCPs
  - General networks
  - Adapt SCP heuristics
  - Mixed results
- This approach
  - Initially examines uniform cost problems
  - Explores efficiently solvable instances
  - Develops formal intuition for new heuristics

MCP on PATHS & TREES

- Path
  - A sequence of *n* nodes connected by (*n-1*) edges
- Tree
  - Graph that contains no cycle
  - Has (*n-1*) edges for *n* nodes
  - Has <u>at least two</u> nodes with degree 1
  - Every two nodes are connected by a unique path
- Independent pair
  - Set of two facilities that cover each other
  - No other facilities cover either one
- Independent trio

THE MCP ON PATHS: UNIFORM EDGE DISTANCES

Optimal Solution

For the MCP on a path with uniform edge distances, an optimal solution exists which consists entirely of independent pairs of facilities and at most one independent trio of facilities.

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THE MCP ON PATHS: UNIFORM EDGE DISTANCES

Elements of the Proof

1) No facilities in groups smaller than two.
2) No facilities in groups larger than three.
3) No more than one independent trio in an optimal solution.

THE MCP ON PATHS: UNIFORM EDGE DISTANCES

Algorithm for Facility Placement:

- Recursive placement of independent pairs
- Finish with an independent pair or independent trio

O(n) to optimality

THE MCP ON PATHS: NON-UNIFORM EDGE DISTANCES

<u>Optimal Solution</u>

For the MCP on a path with non-uniform link distances, an optimal solution exists which consists entirely of independent pairs of facilities and independent trios of facilities. An optimal solution may have more than one independent trio, but will not *require* consecutive independent trios of facilities along the path.

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THE MCP ON PATHS: NON-UNIFORM EDGE DISTANCES

Elements of Proof

1) No facilities in groups smaller than two.
2) No facilities in groups larger than three.
3) An optimal solution *need* not have consecutive independent trios.

THE MCP ON PATHS: NON-UNIFORM EDGE DISTANCES

Simple Path Algorithm

1) Initial Upper Bound.
2) Examine combinations for possible improvement.
3) Enumerate and improve by decrements on incumbent for z*.

$O(e^n)$ to optimality

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THE MCP ON PATHS: NON-UNIFORM EDGE DISTANCES

<u>Bounding the Simple Path Algorithm</u>

Given an initial upper bound, z^0_{UB}, to the optimal solution (z^*) obtained from Step 1 of the Simple Path Algorithm, the strict lower bound on the optimal solution (z^*) is:

$$z^* \geq z_{LB} = z^0_{UB} - \lfloor (z^0_{UB} - t)/6 \rfloor$$

where t=3 if an independent trio is placed during Step 1 of the Simple Path Algorithm; t=0 otherwise.

THE MCP ON PATHS: NON-UNIFORM EDGE DISTANCES

Elements of Lower Bound Proof

1) Facility pattern – all pairs, at most one trio.
2) Minimum number of pairs with possible recomposition and improvement is three.

Significance

➔ At best a 1/6th reduction in the number of facilities that are placed in independent pairs

➔ Termination criterion for Simple Path Algorithm

THE MCP ON PATHS: NON-UNIFORM EDGE DISTANCES

SPA Lower Bound Significance

The upper bound (z^0_{UB}) obtained from Step 1 of the Simple Path Algorithm will result in a feasible solution than is no worse than 20% larger than the optimal solution.

O(n) to within 20% of optimal

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THE MCP ON TREES: BOUNDING THE SOLUTION

Purpose:

- Use tree decomposition and the SPA for the MCP on Paths to bound the MCP on Trees.

Definitions:

- Interior Node $d>1$
- Exterior Node $d=1$
- Tree Decomposition Arc-partition

$$N_P = |N_{d=ODD}|/2$$

$$= (\text{\# of nodes of odd degree}/2)$$

THE MCP ON TREES: BOUNDING THE SOLUTION

Upper Bound

- Sum of the decomposed path solutions

Lower Bound

- Within the MCP framework, the intersection of one path at an interior node with any node of another path will result in a tree with an optimal solution that is between zero and two facilities less than the sum of the independent path optimal solutions.

THE MCP ON TREES: BOUNDING THE SOLUTION

Bounding Effects of Each Node

Node Degree	1	2	3	4	5	6	7	8	9	10	11	12	13	…
Min. No. of Paths through Node	1	1	2	2	3	3	4	4	5	5	6	6	7	…
Maximum Facility Redundancy	0	0	2	2	4	4	6	6	8	8	10	10	12	…

Closed Form Equation

$$\text{MAX } \Delta z_N = 2*(\lceil (d_N/2) \rceil - 1)$$

THE MCP ON TREES: BOUNDING THE SOLUTION

$$\Sigma_P(z_P)^* - \Sigma_N[2*\lceil(d_N/2)\rceil - 2] \quad \leq \quad z_T^* \quad \leq \quad \Sigma(z_P)^*$$

where…

$z_T^* \equiv$ Optimal MCP solution for the tree

$\Sigma_P(z_P)^* \equiv$ Sum of facilities required to solve the MCP problem for the tree when decomposed into the minimum number of paths, N_P.

$\Sigma_N[2*\lceil(d_N/2)\rceil - 2] \equiv$ Sum of maximum facility redundancies over all nodes when conjoining the independent path MCP solutions

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CONCLUSIONS

- MCP has relevant and important applications
- Uniform Edge Distance Paths
  - O(n) to optimality
- Non-uniform Edge Distance Paths
  - O(n) to within 20% of optimal
  - $O(e^n)$ to optimality
- Trees
  - O(n) to strong upper and lower bounds

ONGOING RESEARCH

- Expanded research team
  - Dr. J. Cole Smith, SIE Dept., Univ. of Arizona
  - Dr. Jeff Goldberg, SIE Dept., Univ. of Arizona
- Non-uniform Edge Distance Paths
  - O(n) to optimality
- Variable cost facilities on paths
  - O(n) to optimality

FUTURE RESEARCH

- Capacitated facilities
- Solving instead of bounding the tree solution

QUESTIONS