23 April 2012

Complex Networks: Consensus, Control and Minority Opinion

presented at the

Center for Complex Networks, USMA

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M. Turalska.

Consensus, Control and Minority Opinion

- **What are we going to talk about tonight?**
  - **SCALING RELATIONS**

network property

scaling index

$$Y = aX^{b}$$

size

scaling coefficient

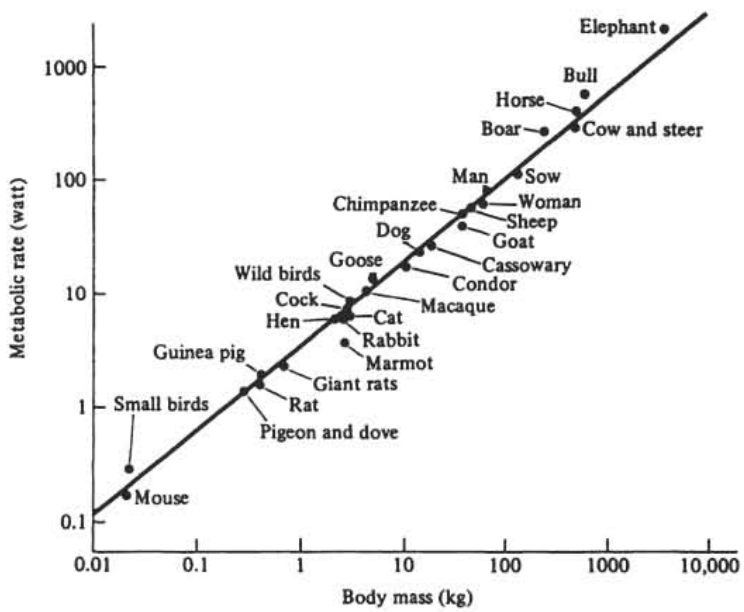
Scaling, fractals, allometry intermittency,….

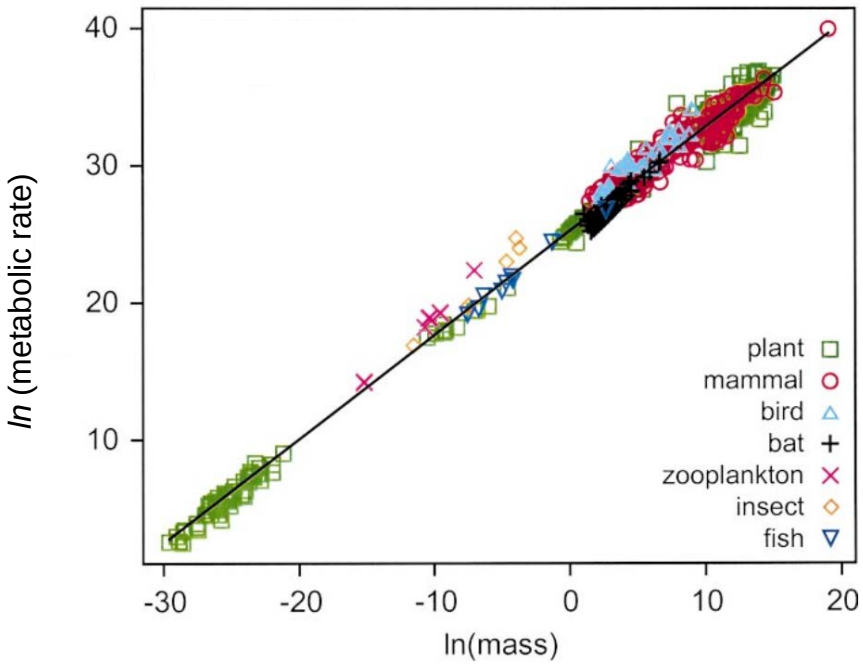
Would you repeat that?

  - Why are scaling relations important?
  - What does scaling have to do with consensus and control?

Consensus, Control and Minority Opinion

- Coarse grained understanding of living networks
  - the functionality is related to the size of the organism
  - allometry relations are entailed by the interactions within the organism



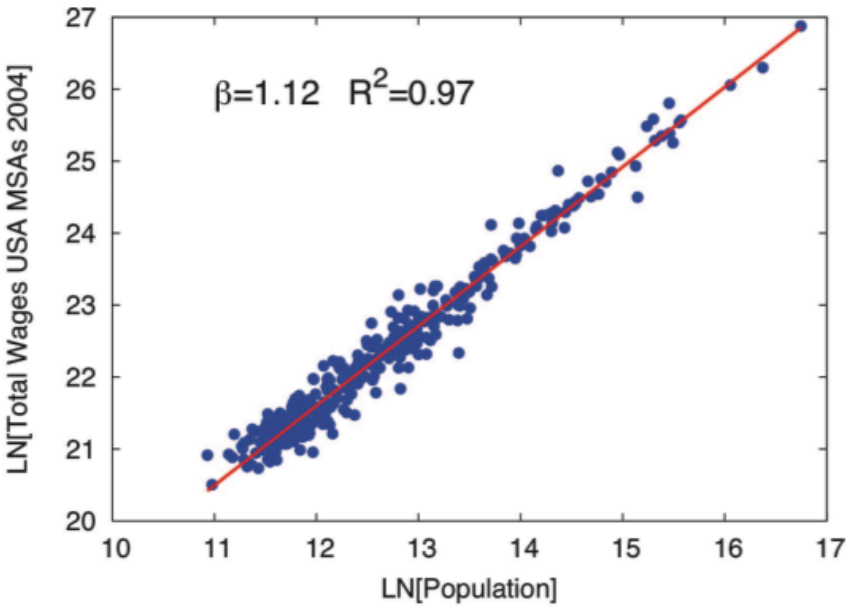


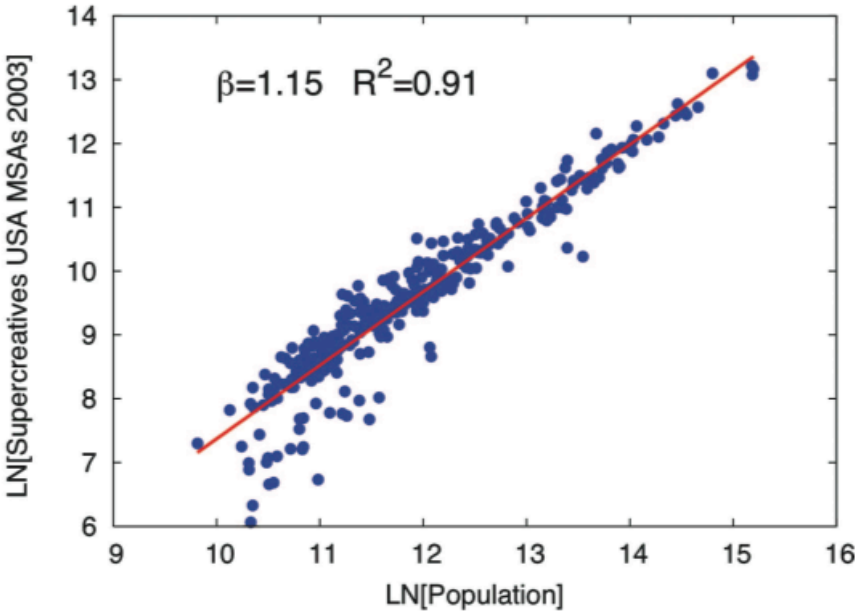
- metabolic rate versus body mass for plants, mammals, fish and insects

$$B = \alpha M^{\beta} \quad ; \quad \beta \approx 0.75$$

Consensus, Control and Minority Opinion

- Scaling also arises in social contexts, very different kinds of interaction
  - increase in wages with city size
  - increase is creativity with population





- Not just a metaphor. $B = \alpha M^{\beta}$

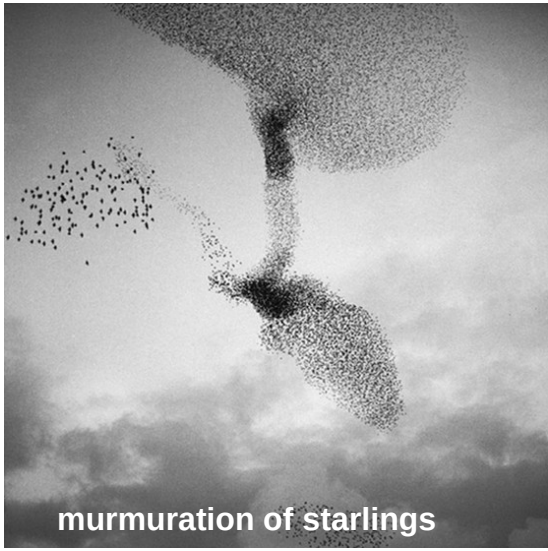
Consensus, Control and Minority Opinion

- What do living systems and social organizations such as cities have in common?

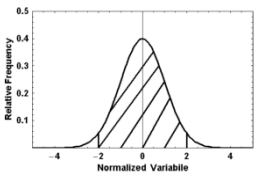
- Each system consist of a network of networks; complex at every level.

- These complex networks entail the scaling manifest in allometry relations.

- So what do we know about networks?

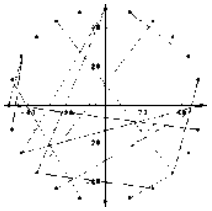


Statistics and networks

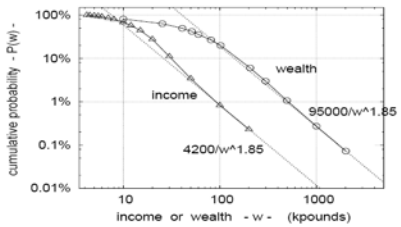


Poisson

Random Networks

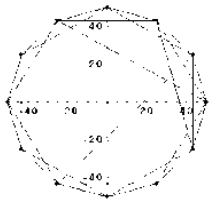


- Erdös-Renyi; random network, 1960s

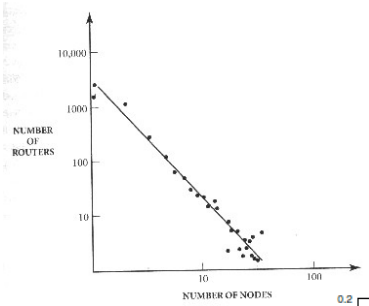


Income distribution

‘Scale-free’ Networks

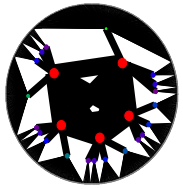


- Watts-Stogatz; Small-world networks, 1990s

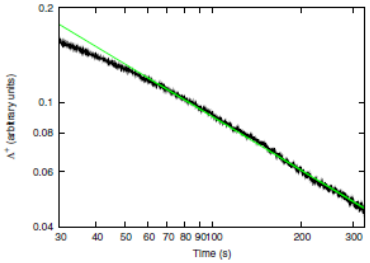


Internet connectivity

‘’Scale-rich’ Networks



- Carlson & Doyle; HOT, 2000s

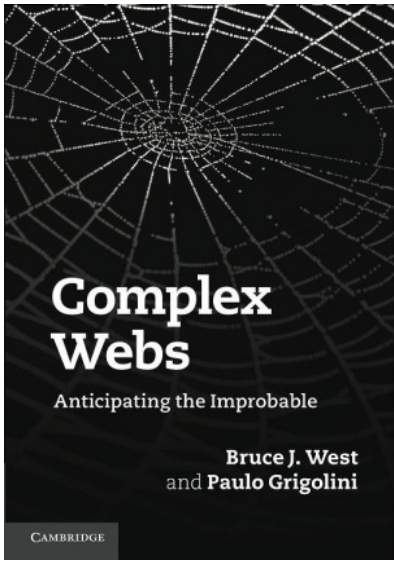


Dynamic Networks

Liquid crystal relaxation of harmonic perturbation

- Grigolini, Turalska, West, 2010s

Some myths about complex networks



- Topology determines network properties
  - e.g., robustness is a consequence of topology

- A unique mechanism produces inverse power-law distributions.
  - Mathews effect, preferential attachment, Yule effect

- Topology determines the degree distribution & the inter-event distribution in time.

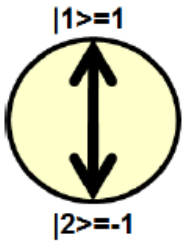
$$P(k) \propto k^{-\nu} \Rightarrow \psi(\tau) \propto \tau^{-\mu}$$

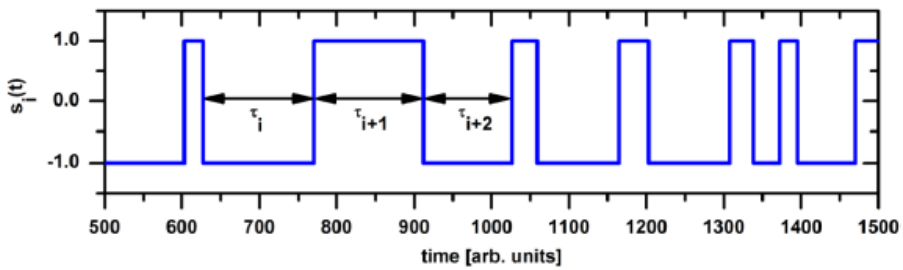
- Dynamics is determined by the topology.

- Inter-network influence is determined by energy gradients.

Two – state Model:
Mean Field Approach

Consensus, Control and Minority Opinion



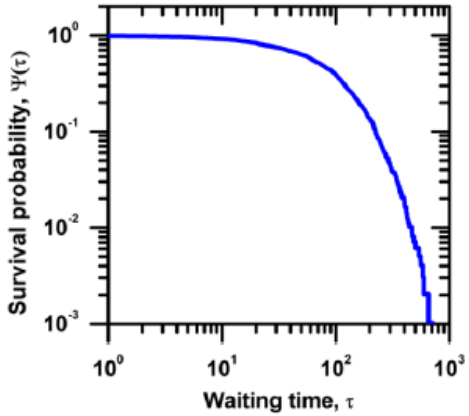


Dynamics of a unit governed by a transition rate *g* is described by two-state maser equation:

$$\frac{d}{dt}p_1 = g(p_2 - p_1)$$
$$\frac{d}{dt}p_2 = g(p_1 - p_2)$$

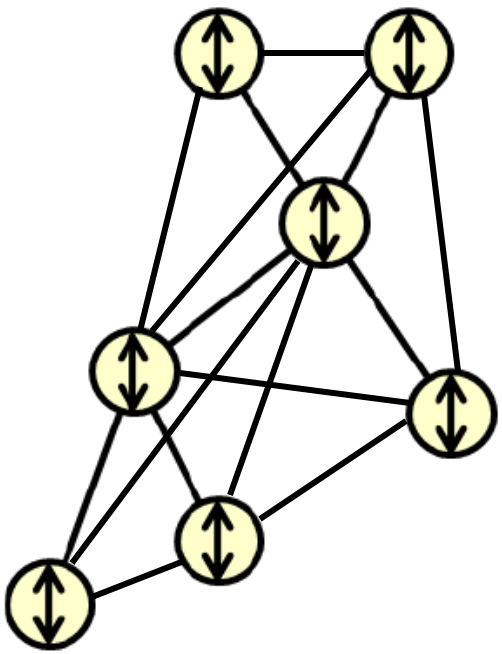
The survival probability function for the sequence of time intervals between consecutive changes of state is defined by an exponential decrease with rate g:

$$\Psi(\tau) = e^{-g\tau}$$



Turalska, Lukovic, West & Grigolini, " Complexity and synchronization", *Phys. Rev. E* **80**, 021110 (2009).

Consensus, Control and Minority Opinion



all-to-all coupling

A unit interacts with all other units to which it is directly connected. Modified master equation is:

$$\frac{d}{dt}p_1 = -g_{12}p_1 + g_{21}p_2$$
$$\frac{d}{dt}p_2 = -g_{21}p_2 + g_{12}p_1$$

where transition rates become time dependent:

$$g_{12}(t) = g e^{\frac{K}{M}(M_1 - M_2)}$$
$$g_{21}(t) = g e^{-\frac{K}{M}(M_1 - M_2)}$$

M - number of nodes connected to given unit

M_j - number of nearest-neighbor nodes in state $|j\rangle$

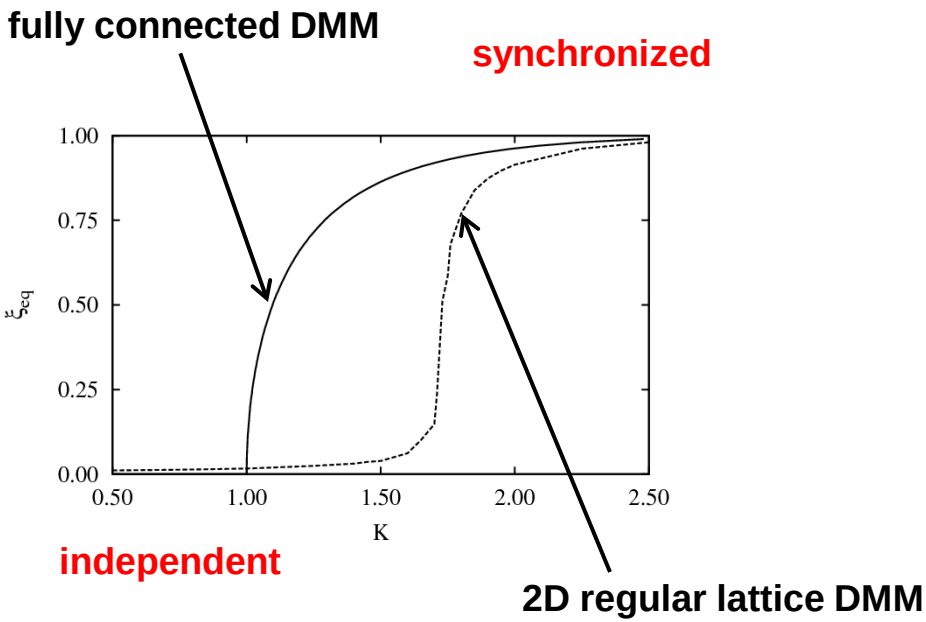
K - control parameter defining the strength of the coupling between interacting units

Consensus, Control and Minority Opinion

• DMM networks undergo phase transitions as the control parameter K is increased. At the critical value $K = K_c$ synchronization between the elements is achieved; consensus is reached.

• The fully connected DMM has the critical value $K_c = 1$.

• The nearest-neighbor DMM on a two-dimensional lattice has the critical value $K_c = 1.7$.



Consensus, Control and Minority Opinion

- For finite *N* with $K \geq K_c$, consensus can be difficult to see, so introduce new variable

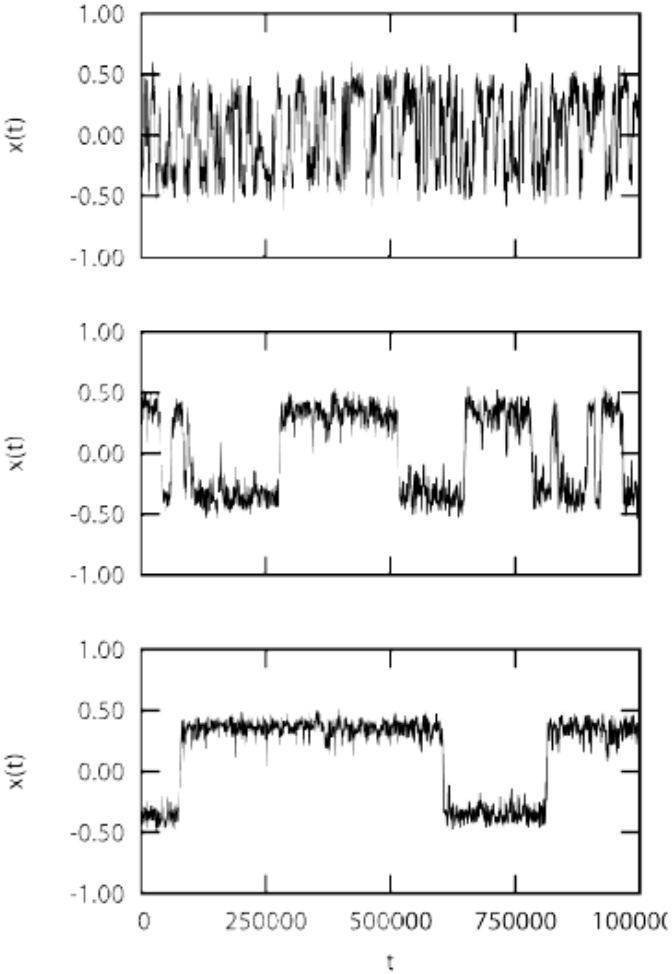
$$\xi(t) = p_1(t) - p_2(t)$$

- The network becomes unstable and Is modeled using a Langevin equation.

$$\frac{d\xi(t)}{dt} = -\frac{\partial V(\xi)}{\partial \xi} + \eta(t)$$

- In the limit $N \to \infty$ or $K \to \infty$ the consensus becomes perfect.

- Fluctuations are produced by finite *N* not a thermal background.



N=100

N=1500

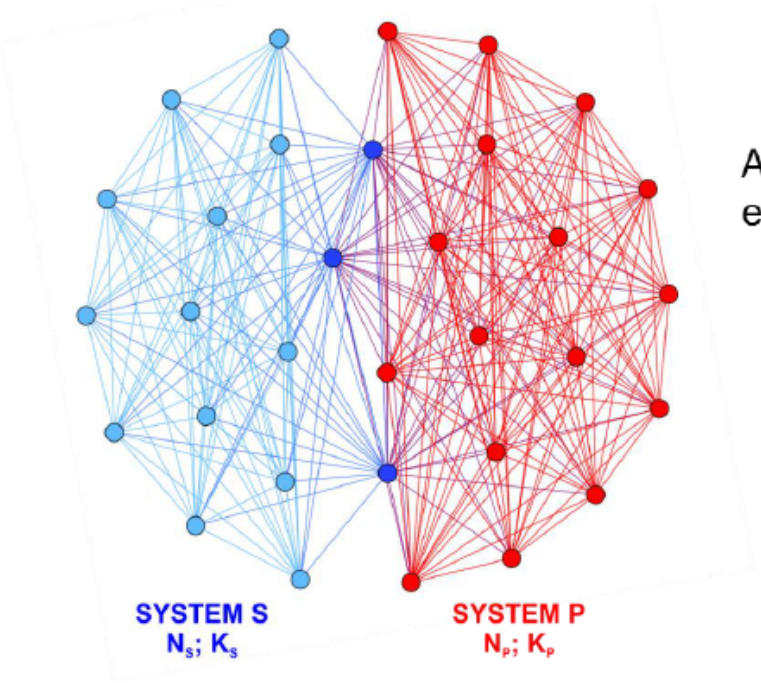
N=2500

Consensus, Control and Minority Opinion

$N_S^* = N_S - f_{N_S}$ nodes of system S evolve as:

$$g_{12}^S(t) = g\exp\left[\frac{K_S}{N_S}\left(N_1^S - N_2^S\right)\right]$$

$$g_{21}^S(t) = g\exp\left[\frac{K_S}{N_S}\left(N_2^S - N_1^S\right)\right]$$



All N_P nodes of system P evolve as:

$$g_{12}^P(t) = g\exp\left[\frac{K_P}{N_P}\left(N_1^P - N_2^P\right)\right]$$

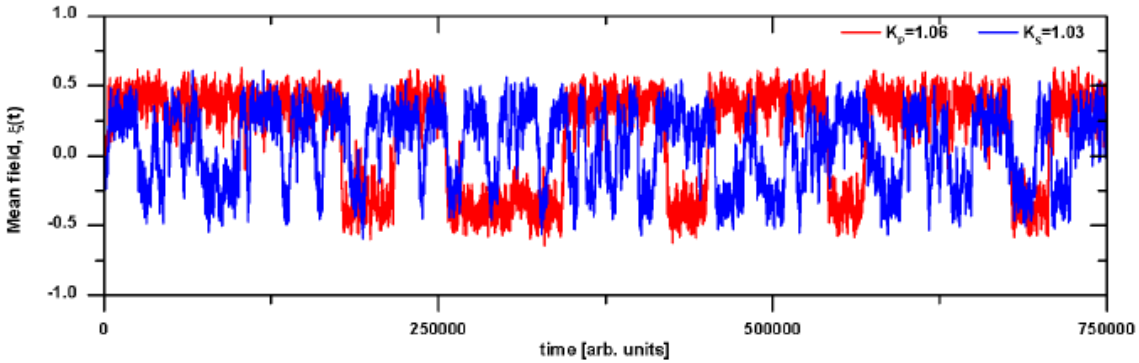
$$g_{21}^P(t) = g\exp\left[\frac{K_P}{N_P}\left(N_2^P - N_1^P\right)\right]$$

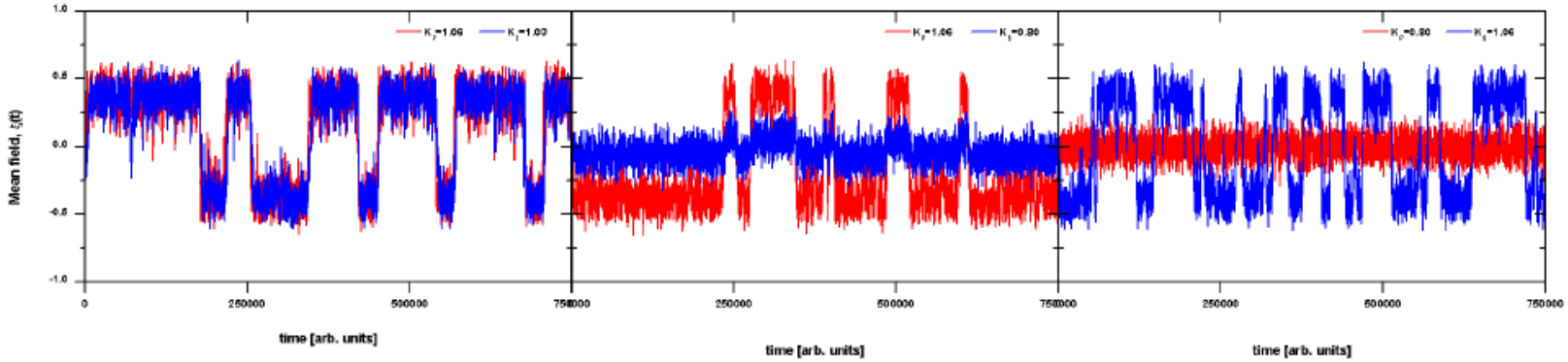
Small fraction f_{N_S} of nodes of system S is linked to all nodes of perturbing system P:

$$g_{12}^S(t) = g\exp\left[\frac{K_S}{N_S}\left(N_1^S - N_2^S\right) + \frac{K_{SP}}{N_P}\left(N_1^P - N_2^P\right)\right]$$

$$g_{21}^S(t) = g\exp\left[\frac{K_S}{N_S}\left(N_2^S - N_1^S\right) + \frac{K_{SP}}{N_P}\left(N_2^P - N_1^P\right)\right]$$

Consensus, Control and Minority Opinion





Introduction of coupling leads to synchronization. $K_S = 1.03$ and $K_P = 1.06$.

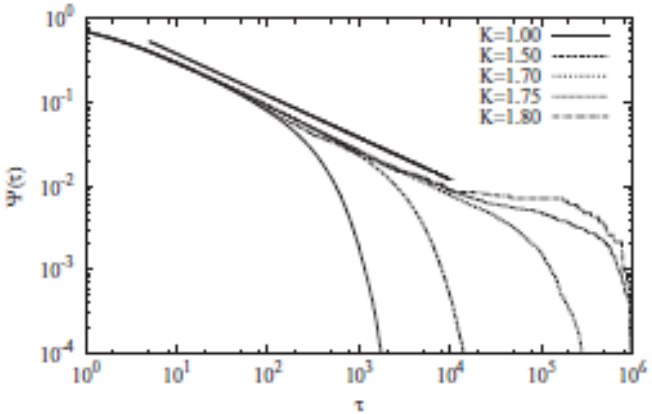
The influence of the perturbation is significantly reduced when system S is not in complex regime; $K_S = 0.80$ and $K_P = 1.06$.

Perturbation that is not complex does not exert an influence on system S; $K_S = 1.06$ and $K_P = 0.80$.

Conclusions - I

• all-to-all coupled network of interacting two-state stochastic elements shows a phase transition with respect to the coupling parameter K.

• when the number of elements is finite the global variable become intermittent and the distribution density of waiting times is inverse power-law with index1.5.



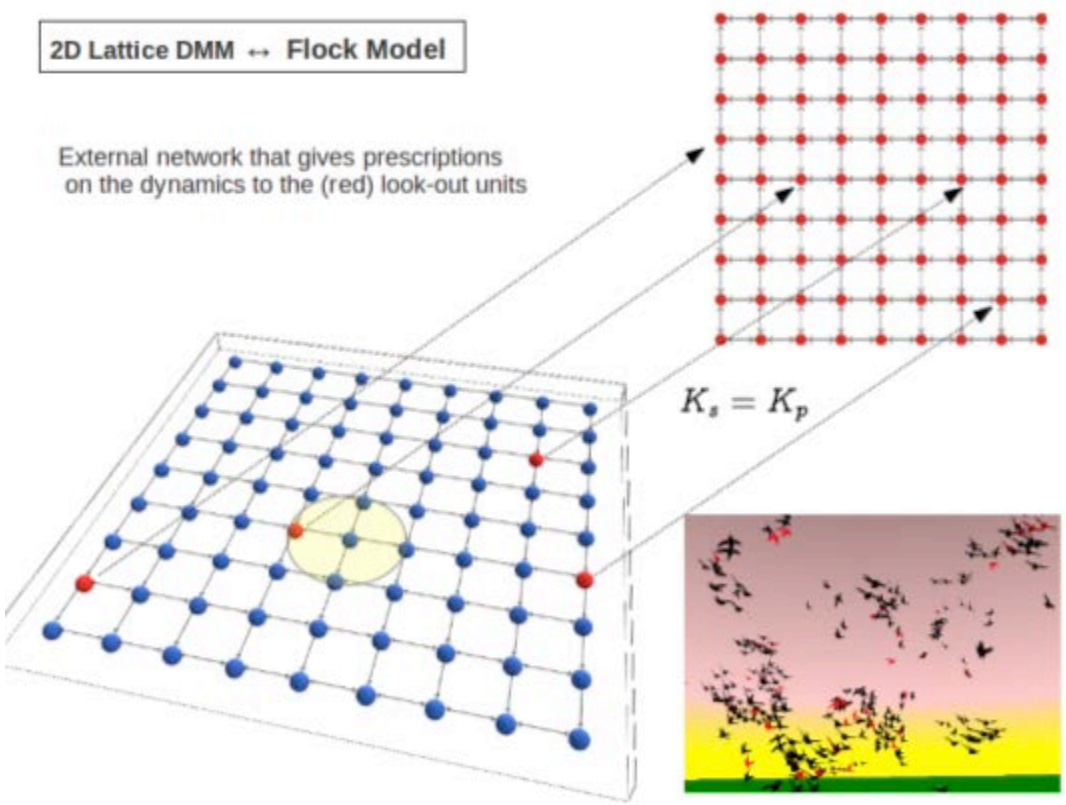
• the sequence of transitions between consecutive consensus states is non-ergodic.

• a network of two-state stochastic elements can be influenced by a perturbing network if both are in a supercritical condition.

Two-state DMM:
Two dimensional lattice

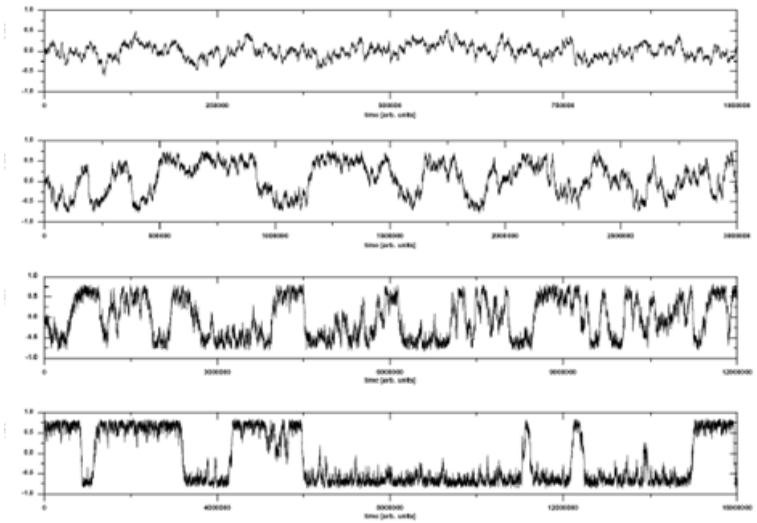
Consensus, Control and Minority Opinion

The individuals in the DMM are distributed on the nodes of a two-dimensional lattice. We study the dynamics of a single complex network and the interaction between complex networks.



The global order parameter is defined as:

$$\xi(t) = \frac{1}{L}\sum_{i=1}^{L} s_i(t)$$



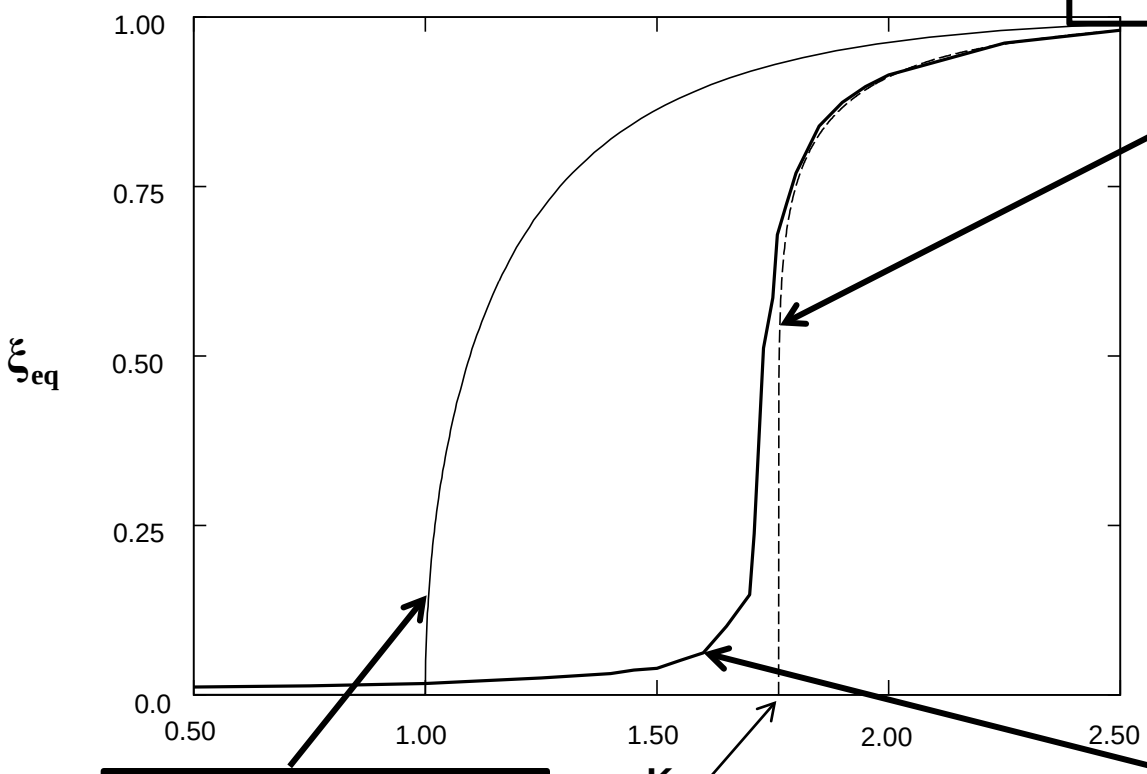
Temporal evolution of the global order parameter $\xi(t)$ on a square lattice of $N = 50 \times 50$ nodes for increasing values of the coupling constant K.

Consensus, Control and Minority Opinion

Phase diagrams

Onsager solution to Ising model

$$\xi_{eq} = \left[1 - \sinh^{-4}(K/2)\right]^{1/8}$$



- Transition from local short-range forces to global long-range forces; water to ice or from a collection of independent individuals to a lynch mob.

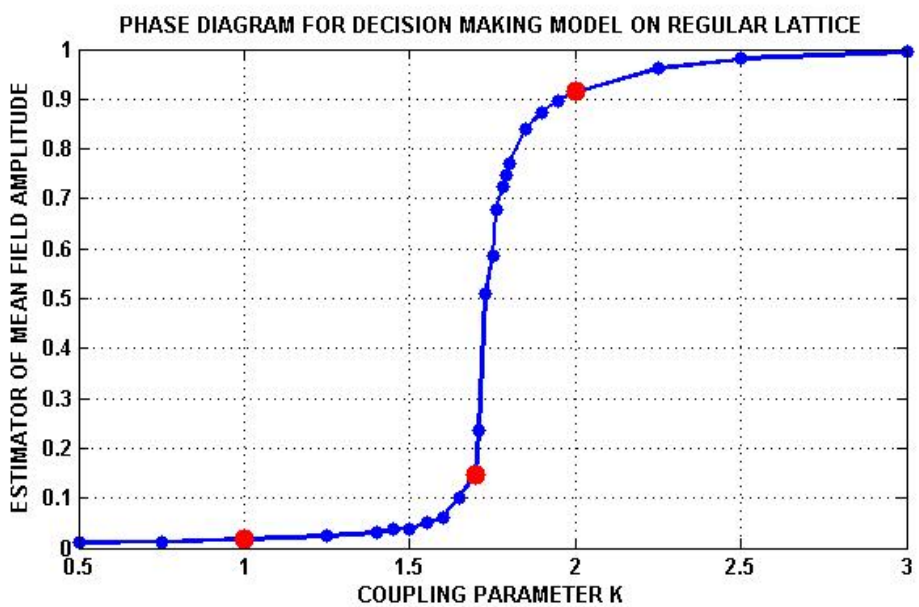
fully connected DMM
$K_C = 1$

$K_C = 2\ln(1+\sqrt{2}) = 1.7627$

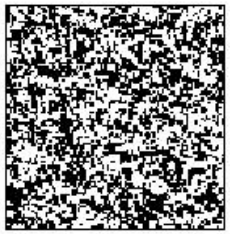
2D regular lattice DMM
$g_0 = 0.01$

Consensus, Control and Minority Opinion

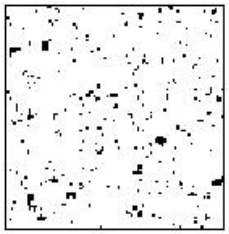
- **DECISION MAKING MODEL – REGULAR 2D LATTICE**
  - **Phase diagram**

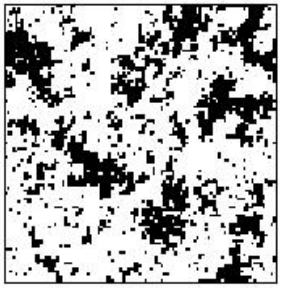


K=1.00



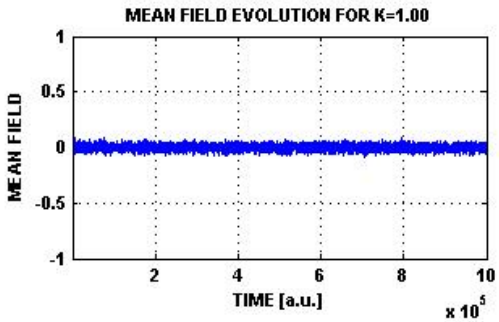
K=2.00

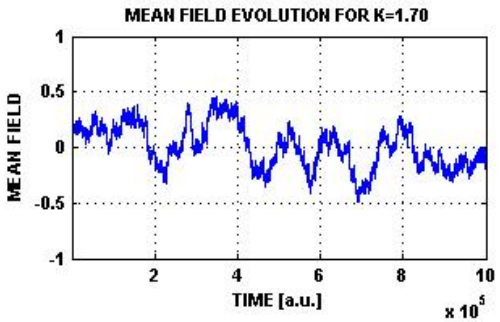


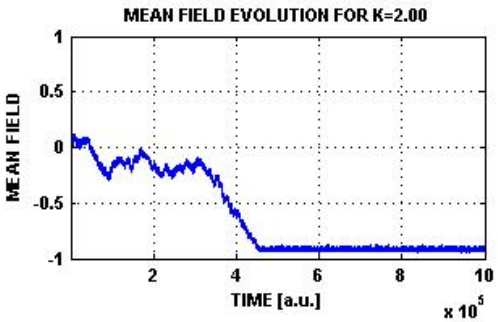


K=1.70

Turallska, West & Grigolini, *Phys. Rev. E* **83**, 061142 (2011)





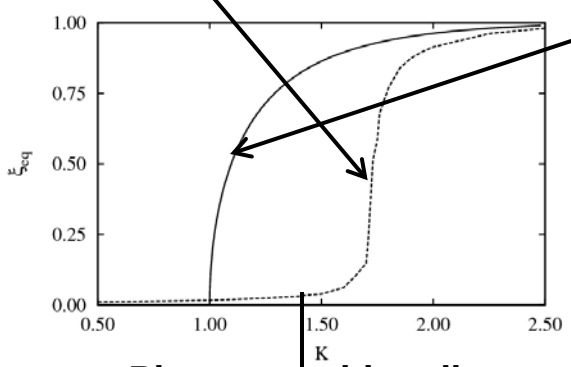


Consensus, Control and Minority Opinion

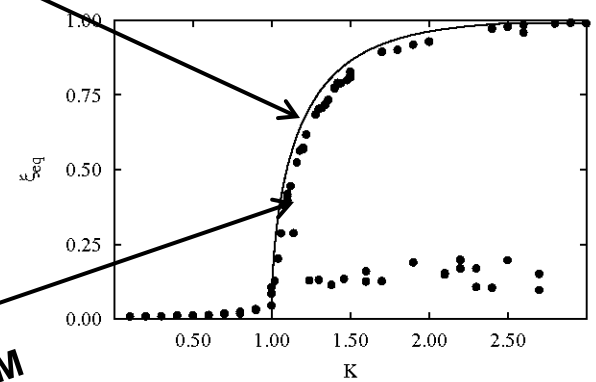
- **DMM dynamics generate scale-free statistics in space and time**

100x100 lattice network

infinite order DMM



Phase transition diagram



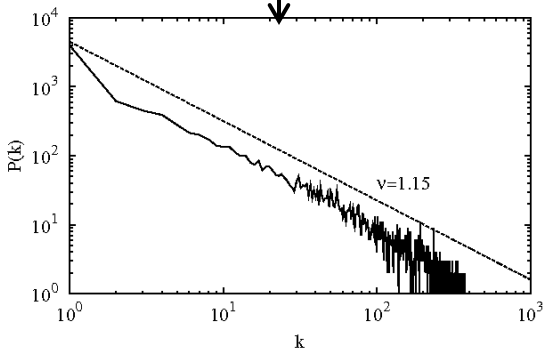
consensus

no consensus

correlated links

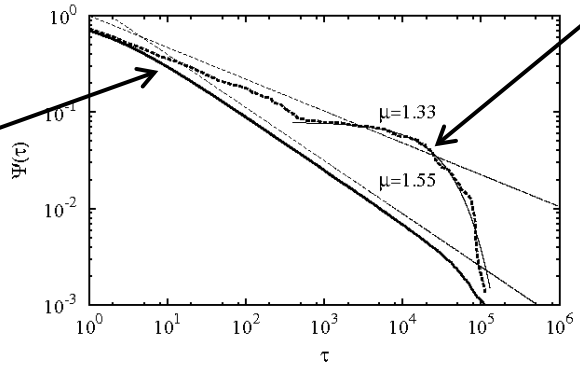
Backbone for DMM

infinite order DMM



Degree distribution for "correlation" DMM

"correlation" DMM



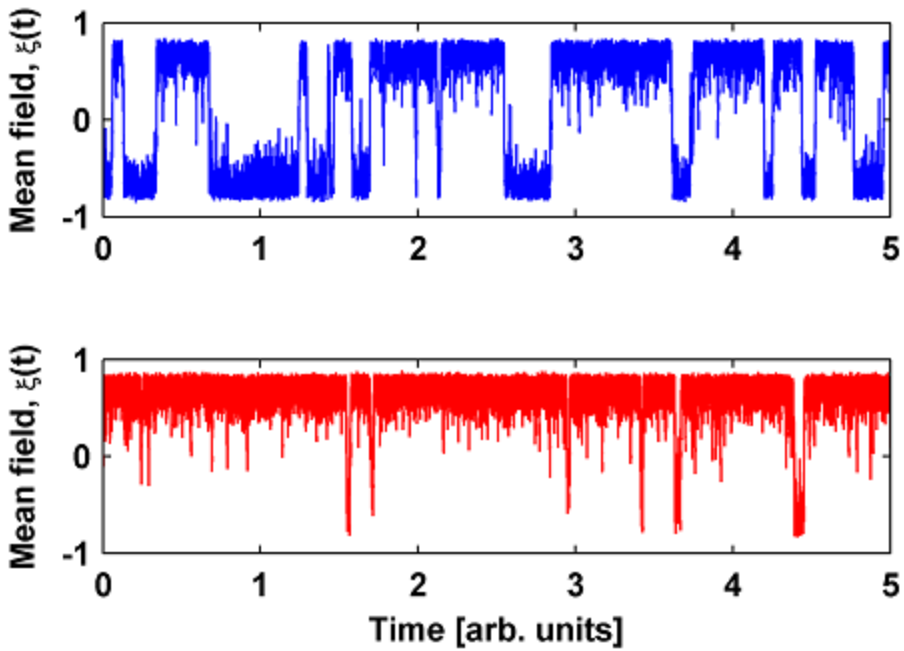
Survival probability vs. time

Conclusions - II

- Two-state model in the phase space of its parameters incorporates the behavior of the two-dimensional Ising model.
- The onset of phase transition corresponds to the fluctuations of global order parameter being characterized by extended temporal complexity.
- The temporal complexity is accompanied by long-range correlations.
- The transitions between organized states occur through intermediate critical state rather than a disorganized phase.

Consensus, Control and Minority Opinion

- Coupling of a committed minority to a DMM in the critical state



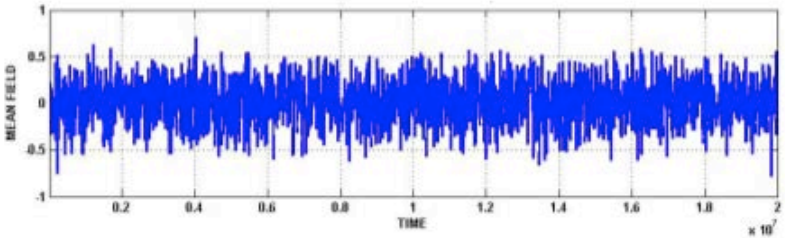
20 × 20 DMM

1% of network replaced with committed minority

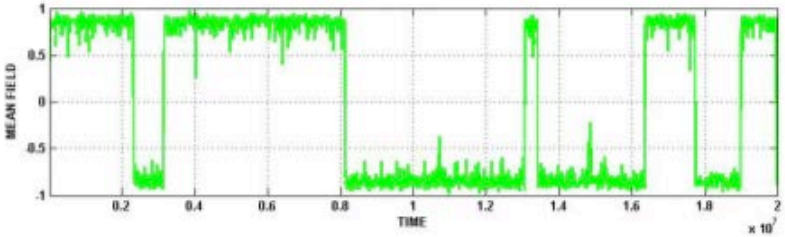
- This is with 1% coupling.

Consensus, Control and Minority Opinion

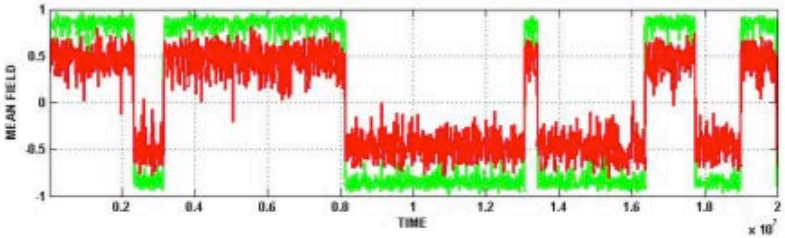
- Two complex networks; individuals from P infiltrating S.



- DMM network S not in critical state



- DMM network P in critical state



- 1% of S replaced with elements of P

Consensus, Control and Minority Opinion

- **General conclusions**

  - Complex networks are described by nonlinear dynamics that undergo phase transitions.

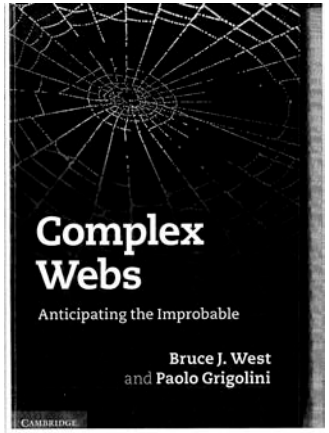
  - The dynamics determine the topology of the scale-free network.

    - The degree distribution is inverse power law with index near 1.0.

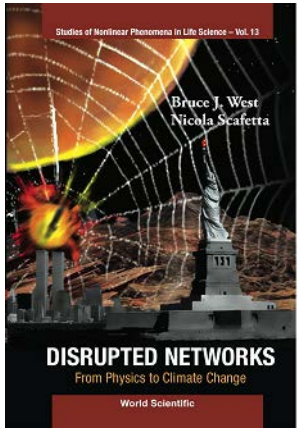
    - The waiting-time distribution is inverse power law with index near 1.5.

    - The degree and waiting-time distributions are independent of one another

2011



2010



Consensus, Control and Minority Opinion