Triangle Points

Finding How A Rule Converges

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October 21, 2017
01:00 EDT

Problem

Let $T_0$ be the interior of a triangle in $\mathbb{R}^2$ with vertices $A$, $B$, $C$. Let $T_1$ be the interior of a triangle whose vertices are the midpoints of the sides of $T_0$, $T_2$ be the interior of a triangle whose vertices are the midpoints of the sides of $T_1$, and so on. Find the points, if any, in the set $\cap_{n=0}^{\infty} T_n$.

For another challenge, prove that you have found all of the points.

Solution

To see what is happening let's draw a random triangle inside the unit square.
\(\text{pts} = \text{RandomReal[{0, 1}, \{3, 2\}];}\)
\[
\begin{align*}
\text{plot} &= \text{Graphics}\left[\{\text{FaceForm[None]}, \text{EdgeForm[Black]}, \right. \\
&\quad \text{Rectangle[]}, \text{EdgeForm[\{Blue, Thick\}]}, \text{Triangle[pts]}\}\right].
\end{align*}
\]

Now we can iterate using the specified rule until the solution converges.

\(\text{end} = \text{Union@Round[FixedPoint[0.5 \left\{\begin{array}{l}
\text{pts }[[1]], \text{pts }[[2]] + \text{pts }[[3]], \\
\text{pts }[[2]], \text{pts }[[3]], \\
\text{pts }[[3]], \text{pts }[[1]], \\
\end{array}\right\} & \text{pts}], 10.0^\times -12]}\)

The solution has converged to a single point. This point is simply the mean of the three points which formed the vertices of the initial triangle.

\(\text{Mean[pts]}\)

\(\text{Out[87]} = \{0.347573, 0.622467\}\)

We plot this red point inside the initial triangle.
We can also attack the problem symbolically. After 5 iterations we have:

\[
\text{Out[89]= } \text{Nest}\left[ \frac{1}{2} \left( \#\{1, 1\} + \#\{2, 1\}, \#\{1, 2\} + \#\{2, 2\} \right),
\quad
\left( \#\{2, 1\} + \#\{3, 1\}, \#\{2, 2\} + \#\{3, 2\} \right), \left( \#\{1, 1\} + \#\{3, 1\}, \#\{1, 2\} + \#\{3, 2\} \right) \right] \&,
\quad
\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}, 5 \right] \text{ // Together}
\]

After 15 iterations we have:

\[
\text{Out[90]= } \text{Nest}\left[ \frac{1}{2} \left( \#\{1, 1\} + \#\{2, 1\}, \#\{1, 2\} + \#\{2, 2\} \right),
\quad
\left( \#\{2, 1\} + \#\{3, 1\}, \#\{2, 2\} + \#\{3, 2\} \right), \left( \#\{1, 1\} + \#\{3, 1\}, \#\{1, 2\} + \#\{3, 2\} \right) \right] \&,
\quad
\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}, 15 \right] \text{ // Together}
\]
the x or y coordinates of the initial triangle's vertices.

\[
\frac{\frac{1}{3} \alpha (2^n + 1) + \frac{1}{3} \beta (2^n + 1) + \gamma \left( \frac{1}{3} (2^n + 1) - 1 \right)}{2^n}
\]

Taking the limit as \( n \to \infty \) this converges to a single point which is the simply the mean of the three points which formed the vertices of the initial triangle.

\[
\text{Out[91]} = \frac{1}{3} (\alpha + \beta + \gamma)
\]

\text{In[91]} = \text{Limit}\left[ \frac{\alpha (2^n + 1) / 3 + \beta (2^n + 1) / 3 + \gamma \left( (2^n + 1) / 3 - 1 \right)}{2^n}, n \to \infty \right]