Modeling in Discrete Dynamical Systems

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Scenario 1: Tanks are Discrete

Consider a pure armor battle (tank vs. tank) between Country X and Country Y. Country X attacks with 200 tanks while Country Y defends with only 65 tanks. After each five-minute period, both sides lose some tanks based upon the number of opposition tanks, the reliability of the their weapons, and the survivability of the tanks. Once a tank from Country X sights an enemy tank, it has a 20% chance of striking the target. Likewise, Country Y tanks are 30% effective. A tank from Country X has a 40% chance of becoming a “kill” if it is hit; Country Y tanks survive hits 65% of the time.

Problem 1: Model this situation using a discrete dynamical model. What assumptions did you make?

Solution 1:

The assumptions are critical to this model! Many different approaches are possible depending on the assumptions made. For example: assume only one tank from country *i* targets a given tank from country *j* at a time. Assume no fratricide. Assume all tanks only acquire one target in each 5-minute period. Assume a “kill” means the tank no longer participates in the battle in any way. Further, assume that if country X has more tanks than country Y, they don’t fire more tanks than targets available. This last assumption is critical. Take the first 5-minute period: Country X has 200 tanks. If all 200 acquire targets, they may fire at the same tank or a tank already killed. We assume each of the 200 waits their turn to acquire a live target! Is this realistic? Probably not, but without this assumption modeling this situation is very difficult.

Let’s examine a possible model:

Let $X(n)$ = the number of country X tanks remaining after n 5-minute periods
$Y(n)$ = the number “ Y “

$X(0) = 200 \qquad Y(0) = 65$

$$X(n+1) = X(n) - 0.4(0.3)Y(n)$$
$$Y(n+1) = Y(n) - 0.35(0.2)X(n)$$

Problem 2: If a country surrenders when they have 5 or fewer tanks, which country will win? How long will the battle take? How many tanks does the losing country need to start with to change the outcome?

Solution 2:

Iterating the previous model in Mathcad shows that country X wins after 5 five-minute intervals:

$$:= \begin{bmatrix} \\ \\ \end{bmatrix} \qquad :=$$

$$+ \quad \begin{bmatrix} & - \\ - & \end{bmatrix}$$

Next, we can “play with” the initial conditions for country Y to determine that the outcome will change by starting Country Y with 153 tanks:

$$A_0 := \begin{bmatrix} 200 \\ 153 \end{bmatrix} \qquad n := 0..\ 50$$

$$A_{n+1} \quad \begin{bmatrix} 1 & -.12 \\ -.07 & 1 \end{bmatrix} \cdot A_n$$

Problem 3: In order to brief your results, you must prepare for questions about the assumptions you made and their impact. Choose one assumption and discuss how it effects your results.

Solution 3:

Again, many assumptions substantially change the model (perhaps to make it too complex for our purposes). An easy one is to consider is fratricide. We can add a fratricide term as follows:

$$X(n+1) = 0.99X(n) - 0.4(0.3)Y(n)$$
$$Y(n+1) = 0.94Y(n) - 0.35(0.2)X(n)$$

Other assumptions to discuss: tanks acquiring the same target would reduce the kill rates; hit tanks might still fire effectively which increases the kill rate for the enemy…

Scenario 2: The Motor Officer

You are the Squadron Motor Officer (SMO) for the “Thunder” Squadron of the 3rd Armored Cavalry Regiment. You realize that the Squadron is losing 5% of the operationally ready Cavalry Fighting Vehicles (CFVs) each week. There are 72 CFVs currently in the Squadron. The Squadron Commander is obviously concerned about this trend and asks you to tell him how many CFV’s he’ll still have operational after 8 weeks. The squadron needs fifty-six (56) B/CFVs to be considered combat effective.

Problem 1: What simplifying assumptions are you going to make in order to model this situation?

Solution 1:

> We need to assume the rate does not change, that we don’t get more tanks, fix broken ones, etc. There is also the troubling issue of how to handle fractional vehicles. We can assume that we can ignore these at each step or at the end of the computation.

Problem 2: Develop a mathematical model for this situation. Will the squadron still be combat effective?

Solution 2:

> A simple dynamical system model: Let $a(n)$ = the number of CFVs after n weeks ($n = 0,1,\ldots$).
>
> We find $a(0) = 72$ and $a(n+1) = 0.95\, a(n)$.
>
> The solution function is $a(k) = 72(0.95)^k$. Therefore, $a(8) = 72(0.95)^8 = 47.7$
>
> We won’t be combat effective.

Problem 3: Because of your advice on the CFV issue, the Squadron Commander has managed to get a shipment of 3 new CFV’s each week to help replace the 5% that you’re still losing. Model this as a DDS and tell the Squadron Commander how many CFV’s will be available after 8 weeks.

Solution 3:

The new model is: $a(n+1) = 0.95a(n) + 3$.

A particular solution is the equilibrium value of $a_e = 60$.

Combining this with the homogeneous solutions gives the general solution:

$a(k) = c(0.95)^k + 60$

To satisfy the initial value, $c = 12$ giving: $a(k) = 12(0.95)^k + 60$

Now, $a(8)$ is 67.96, and we are effective.

Problem 4: Are the assumptions you made reasonable? Discuss how the results of your model would change if any of the assumptions were incorrect. How would you address this in a briefing to your Squadron Commander?

It is hard to answer the “fractional tank” issue. Suppose after one week the model says we have 72.4 tanks; we probably ought to not count the fractional part. Thus, after each iteration we might best stop, truncate, and then continue. If this is the way to go, the current results are too optimistic! This is clearly something to brief the commander.

The rate is hard to comment about. It would be nice to have historical data to analyze.

To brief this, it might be nice to run the model with some different rates and assumptions about fractional vehicles to give the commander some different perspectives on what might happen.