# Parachute Jumping, Falling, and Landing

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Introduction

It is extremely important that leaders of airborne units understand the safety, medical, and operational issues involved in parachute jumping. Mathematical modeling of the flight and the landing of either a person or a cargo package can help with this understanding. In this section, we describe, analyze, and model several components of parachute jumping. These are basic models using simplified data in order to begin the study of these issues and should not be used for actual airborne operations. The Army has numerous technical manuals that govern the technical issues associated with airborne operations.

The Situation
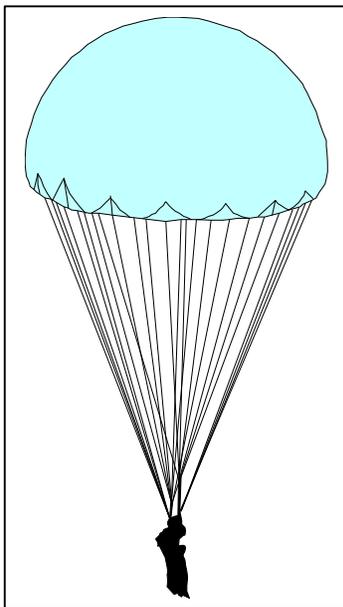
An airborne jumper, whose mass is 103 kg, leaps with a velocity of 0.555 m/s straight out (direction of the *x*-axis) from the side of an airplane which is moving at a velocity of 115 m/s (direction of the *y*-axis). The airplane is flying at an altitude (height on the *z*-axis) of 4000 meters. When the jumper leaves the aircraft, he is high enough in altitude that the air resistance can be considered negligible, and he falls as a freely falling body for 11.5 seconds until he pulls the parachute cord. After that point, the fall becomes a three dimensional projectile motion problem with *non*-constant acceleration along each axis as air resistance must now be taken into account. The reference axes are established as follows (*x*-axis in direction of jumper leaving plane, *y*-axis in direction of plane when the jumper exits, *z*-axis is vertical, and the origin is on the ground directly under the plane the moment the jumper exits).

When the parachute cord is pulled, we assume that the parachute deploys immediately and exerts a combined drag and air resistance force of magnitude $D = dv^2$, where *d* is the drag coefficient due to the parachute and air resistance, and *v* is the velocity of the jumper. In this case, we use $d = 20$, but normally the value of *d* is difficult to determine, since it depends on so many variables.   Assume the drag force is exerted only along the vertical direction (z-axis), and that the effect of air resistance along the y-direction can be modeled as a force of magnitude $F = -bv$, where the minus sign indicates that the force opposes the

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direction of motion and b is the drag coefficient due to air resistance. In this case, we use $b = 10$.

Simultaneous to the opening of the parachute, the jumper experiences a cross wind in the positive x-direction of 1.2 m/s. The force due to this cross wind can be modeled as $G = b(w - v)$, where b is the drag coefficient due to air resistance (b is set to 10 above), v is the speed of the jumper, and w is the cross wind speed.

The collision between the jumper and the ground can also be modeled using mathematics and physics. There are several medical and safety factors that need to be considered for such a collision. Rapid deceleration of the head can be very serious due to the sheer strain on the brain stem (shear means that the *shape* of an object has changed as the result of forces). A measure of this shear strain to the head is the severity index I, which is determined by the equation

$$I = [2(v) / (g \, \Delta t)]^{2.5} (\Delta t),$$

where v is the velocity at impact, Δt is the duration of impact, and g is the acceleration due to gravity. When the severity index for a collision is above 1000, the collision is fatal. When the value of the severity index is approximately 400, unconsciousness and mild concussion are the result.

The stress on the long bones of the legs during the collision is compressive. When the compression force per unit area exceeds the ultimate tensile strength of the bone (given below for various bones, with their associated cross-sectional areas), the bone breaks in compression.

| Bone	Ultimate Tensile Strength (N/m^2)	Average Cross-Sectional Area (m^2)
Femur	1.21×10^8	5.81×10^{-4}
Tibia	1.40×10^8	3.23×10^{-4}
Spinal Cord (back)	2.20×10^8	4.42×10^{-4}
Spinal Cord (neck)	1.80×10^8	4.42×10^{-4}

Table 1

The compressive force F experienced as a result of the impact on the earth can be determined from a physics model using momentum. Momentum p is defined as $p = mv$ and the Force model is $F = \Delta p / \Delta t$, where Δp is the change in momentum and Δt is the time duration of the collision.

If the jumper lands on the ground, we assume that he performs a perfect "parachute landing fall," as he was taught in his free fall class. The time of his impact will depend upon how soft the ground is where he lands. Snow or soft sand, for example, would significantly extend his duration of impact and thus reduce the force of impact.

More Simplifying Assumptions

For our discussion and the initial modeling of this situation, we use the following assumptions in addition to or amplifying those already discussed:
- We are not considering the complexities of landing such as landing on uneven ground or in the trees or in an inappropriate manner.
- We assume the parachute opens instantaneously and the drag force $D = dv^2$ takes effect immediately with no period of transition.
- The jumper does not bend or twist upon landing, and thus the breaking torques of various bones and ligaments are not being considered.
- The jumper leaves the airplane by jumping straight out with no rotation.
- The information given for the severity index and the bone compression values for the tibia and femur are correct (although in reality they are approximate).
- Even though the forces needed to provide the deceleration of the body produce great tensile stresses on ligaments and tendons and the failure level for these are much lower than for bone, we ignore these effects and concentrate on the bones.
- The impact force is transmitted to the bones without attenuation through the ligaments or muscles.
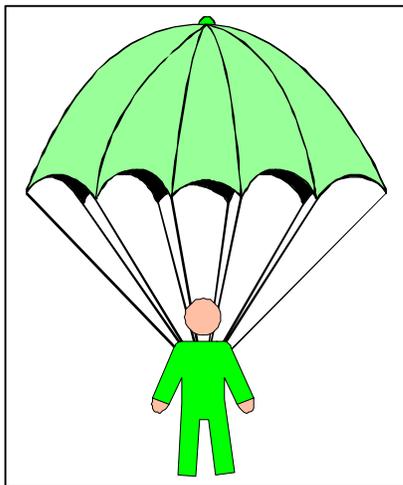
Time and Path of the Fall

The jumper waits 11.5 seconds before pulling his parachute open. During this period of time, he is in free fall (no air resistance). His initial velocity in the vertical direction (z-axis) is zero, and the acceleration is due only to the gravitational force of -9.80 m/s^2. Thus, the jumper's vertical position at the time he deploys his parachute is given by $z(t) = z_0 + v_{z0}t + 0.5\, a_z t^2$. Substitution produces $z(11.5) = 4000 + 0.5\,(-9.80)\,(11.5)^2 = 3351.975$ m.

His initial velocity in the x-direction is 0.555 m/s, and there is no acceleration in this direction (air resistance is negligible during the first 11.5

seconds). So, his position in the x-direction is given by $x(t) = x_0 + v_{x0}t + 0.5\,a_x t^2$. Substitution produces $x(11.5) = 0 + 0.555(11.5) + 0 = 6.3825$ m.

Similarly, the initial velocity in the y-direction is 115 m/s, and the acceleration in this direction is zero. Thus, the position equation in the y-direction is $y = y_0 + v_{y0}t + 0.5\,a_y t^2$. Substitution produces $y(11.5) = 0 + 115(11.5) + 0 = 1322.5$ m.

Therefore, the position coordinate at which the jumper opened his parachute is $(x, y, z) = (6.3825, 1322.5, 3351.975)$ m.

When the jumper pulls the parachute rip-cord, his velocity in the x- and y-directions will be the same as the initial values for those velocities when he leaped from the aircraft. His vertical velocity can be determined by the model $v_z = v_{z0} + a_z t$. Substitution produces $v_z = 0 + (-9.8)(11.5) = -112.7$ m/s.

Therefore, the velocity of the jumper when he pulls the parachute rip-cord is given by $(v_x, v_y, v_z) = (0.555, 115, -112.7)$ m/s.

There are several ways to model the first few seconds of the parachute deployment. First, we have assumed that the chute deployment is instantaneous. In general, the jumper is falling very fast during the final phase of free fall and slows considerably to a constant (or terminal) velocity over some interval of time. For our situation, we will assume, based on an experiment, that this transition phase lasts 3 seconds and that during this transition, the jumper falls 53 meters downward.

Now we determine the terminal (constant) velocity of the jumper when the parachute is deployed. He reaches terminal velocity when the drag force of the parachute, $D = dv^2$, is equal to his weight, mg, so that the resultant acceleration is zero. In other words, when $dv^2 + mg = 0$ the terminal (constant) velocity is reached. Solving for v, we get $v = \sqrt{-mg/d}$. Substitution produces,

$v = \sqrt{-105(-9.8)/20} = 7.1042$ m/s in the downward direction.

Since at terminal velocity the jumper is falling downward at a constant rate of 7.1042 m/s, the time required to fall the remaining distance of 3298 meters is: $(3298)/(7.1042) = 464.2$ seconds. The total time of the fall is calculated by summing the time spent in free fall motion and the time with the parachute open, both before and after terminal velocity was reached. Then the total time equals $11.5 + 3.0 + 464.2 = 478.7$ seconds.

Determining the Landing Coordinates

Applying Newton's Second Law to the jumper's motion after the parachute deploys in both the x and y directions, we obtain the following models for the equations of motion.

x-direction: $F_x = ma_x$

where $\qquad F = b(w - v)$

so, $\qquad b(w-v) = ma_x$
$bw - bv = ma$
$ma + bv = bw$
$mx'' + bx' = bw$
$103x'' + 10x' = 12.$

y-direction: $F_y = ma_y$

where $\qquad F = -bv$

so, $\qquad -bv = ma_y$
$ma + bv = 0$
$my'' + by' = 0$
$103\,y'' + 10y' = 0\ .$

Where m is the mass of the jumper, b is the coefficient of air resistance, and w is the wind speed. We solve these equations to obtain the position function. The initial conditions are the position and the velocity values at the time the parachute was opened.

y-direction: Use the characteristic equation $mr^2 + br = 0$, with roots $r = 0$ and $r = -(b/m)$. So, the form of the general solution is $y(t) = C_1 e^0 + C_2 e^{-(bt/m)}$, where constants C_1 and C_2 are found by using the initial conditions.

These conditions are at time $t = 0$; the time at which the parachute is opened and air resistance is no longer negligible. So at this time, $y = 1322.5$ m and $v_y = 115$ m/s (from above). Substitution produces: $1322.5 = C_1 + C_2$ and $115 = -(b/m) C_2$, where $b = 10$ kg/m and $m = 103$ kg. Therefore, $C_2 = -1184.5$ and $C_1 = 2507$. The solution is written as $y(t) = 2507 - 1184.5\ e^{-(10/103)t}$.

x-direction: In the x-direction, the differential equation is non-homogeneous, so the associated homogeneous equation $mx'' + bx' = 0$ should be solved first, and then the non-homogeneous equation $mx'' + bx' = bw$ is solved. The characteristic equation for the homogeneous differential equation is $mr^2 + br = 0$ where the roots are: $r = 0$ and $r = -(b/m)$. So, the complementary solution is: $x_c = C_1 e^0 + C_2 e^{-(bt/m)}$. Now we solve for the particular solution, x_p, of the non-

homogeneous equation $mx'' + bx' = bw$. We conjecture $x_p = At$ (the conjecture of $x_p = A$ does not work). Then, $x_p' = A$, and $x_p'' = 0$. Substituting this solution into the differential equation yields $m(0) + bA = bw$. Therefore, $A = w$ and $x_p = wt$.

The general solution is $x = x_c + x_p$ Therefore, for our non-homogeneous differential equation, the solution is $x = C_1 + C_2\, e^{-(bt/m)} + wt$, and substitution of our parameter values produces $x(t) = C_1 + C_2\, e^{-(10/103)t} + 1.2\, t$.

We use the initial condition to determine the constants C_1 and C_2. Note that again these are the conditions at time $t = 0$, that is, the time at which the parachute opened and air resistance is no longer negligible. So, $x(0) = 6.3825$ m and $v_x(0) = 0.555$ m/s (from above). This substitution produces $6.3825 = C_1 + C_2$ and $0.555 = -(b/m) C_2 + w$. So, $C_2 = 6.6435$ and $C_1 = -0.261$. We obtain the solution $x(t) = -0.261 + 6.6435\, e^{-(10/103)t} + 1.2\, t$.

In order to find the position coordinates of the jumper's landing site, we use the position equations and the total time of the jump. Given the position equations found above, simply substitute into these equations the value of the time when the jumper lands (the total time experienced by the man-parachute system), $t = 467.2$ seconds. We find the landing position is: $x = 560.4$ m, $y = 2507$ m, and $z = 0$ m.

The Landing

We need to determine the jumper's final velocity (i.e., his velocity immediately before impact) and the force of the collision upon the jumper. The velocity equations will be the derivatives, with respect to time, of the position equations. Therefore, we compute

y-direction: position equation ---> $y(t) = 2507 - 1184.5\, e^{-(10/103)t}$

velocity equation ---> $v_y(t) = (-1184.5)(-10/103)\, e^{-(10/103)t}$

$= 115\ e^{-(10/103)t}$

x-direction: position equation ---> $\quad x(t) = -0.261 + 6.6435\ e^{-(10/103)t} + 1.2\ t$

velocity equation ---> $\quad v_x(t) = (6.6435)(-10/103)\ e^{-(10/103)t} + 1.2$

$$= -0.645\ e^{-(10/103)t} + 1.2$$

At the time of impact, the jumper has reached terminal velocity in the *z*-direction. Therefore, we know $v_z(t) = -7.1042$ m/s. Since we determined the total time of the fall is 478.7 seconds, and he opened his parachute at 11.5 seconds, we are interested in his velocity at 467.2 seconds. This is the time value we substitute into the velocity equations above to obtain the velocity upon impact in each direction:

$$v_x(467.2) = -0.645\ e^{-(10/103)(467.2)} + 1.2 = 1.20 \text{ m/s}$$

$$v_y(467.2) = 115\ e^{-(10/103)(467.2)} = 0.0 \text{ m/s}$$

$$v_z(467.2) = -7.1042 \text{ m/s}$$

In vector notation, we write, $v = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, so $v = (1.20\ \mathbf{i} + 0\ \mathbf{j} - 7.1042\ \mathbf{k})$ m/s.

The magnitude of the velocity vector gives the speed of the jumper at the time of impact. In order to determine the speed we use the Pythagorean theorem:

$$\text{speed} = \sqrt{(v_x^2 + v_y^2 + v_z^2)} = \sqrt{(1.2)^2 + (0)^2 + (-7.1042)^2} = 7.2048 \text{ m/s.}$$

When the jumper collides with the ground he experiences a change in momentum Δp, given by $p_{final} - p_{initial}$. After the jumper collides with the ground, he has no momentum. Therefore, $p_{final} = 0$. Additionally, if we assume that the collision takes place along a single direction, we can use the speed on impact to determine $p_{initial}$. Momentum is given by $p = mv$, where *m* is the mass of the jumper and *v* is his velocity at impact. We can determine the initial momentum as $p_{initial} = mv = (103)(7.2048) = 742.0944$ kg·m/s.

Since the final momentum is 0, the change in momentum Δp is $0 - 742.0944 = -742.0944$ kg·m/s. This can also be described as 742.0944 N·s, where N = 1 kg·m/s^2 indicates force in units of Newtons.

We determine the force of impact using the equation $F = \Delta p / \Delta t$. If we assume the jumper landed in thick bushes with a time of impact of 200 milliseconds (0.200 seconds), we calculate $F = (742.0944) / (.200) = 3{,}710.472$ N.

Injuries?

From the results we have calculated, we can determine if the jumper sustained any injuries. Table 1 gave us the ultimate tensile strength in N/m^2 and the average cross-sectional area in m^2 for four bones vulnerable to injury in this type of fall. By multiplying the tensile strength and the cross section we obtain the maximum force each bone is able to sustain without breaking. Table 2 provides this information.

Bone	Ultimate Tensile Strength (N/m^2)	Average Cross-Sectional Area (m^2)	Max Sustainable Force (N)
Femur	1.21×10^8	5.81×10^{-4}	70301
Tibia	1.40×10^8	3.23×10^{-4}	45220
Spinal Cord (back)	2.20×10^8	4.42×10^{-4}	97240
Spinal Cord (neck)	1.80×10^8	4.42×10^{-4}	79560

Table 2

From Table 2 we see that a force of 3,710.47 N would not break any of the four bones analyzed.

Finally, we check for the possibility of injury to the head by shear strain. We use the severity index, I, to determine possible injury. We calculate $I = [2(v)/(g \, \Delta t)]^{2.5} \, (\Delta t) = [2(7.2048) / [(9.8)(.200)] \,]^{2.5} \, (.200) = 29.310$. The severity index of 29.310 is not close to the concussion level of 400. Therefore, we conclude that there should be no serious injuries to the head caused by this landing.

Conclusion

We have used mathematical modeling in a parachute jump and landing scenario to analyze some of the factors associated with such an activity.   Certainly, more detailed and sophisticated analyses could be performed to understand more of the components of parachute jumping. There are many issues that this scenario has ignored or that were assumed to hold that could use further study. For instance, the transition period from free fall to steady-state falling with a deployed chute contains numerous factors that are worthy of more study (see the exercises). Also, parachute deployment of heavy equipment like vehicles, weapon systems, and supplies is also worthy of study to insure the cargo lands safely and in the proper location.

Exercises

1.   Are there additional assumptions being made that should be explicitly listed? What are they? Write a short essay that describes the role assumptions play in the modeling process.

2. Use the severity Index formula to determine the shear strain on the head during the 3 second transition period from free fall to the chute-deployed steady-state constant velocity.

3. What would happen if the jumper in the scenario landed on a very hard surface, like rocks, and the interval of impact was reduced to $\Delta t = 0.01$ seconds? Do you predict any injuries for the jumper?

4. Solve for the landing time and location, if the same scenario and model are used with an exit altitude of 6,000 meters.

5. Analyze the injury situation, if the same scenario and model are used with an exit altitude of 6,000 meters.

6. In this scenario the transition phase from free-fall to steady-state descent with a fully deployed chute is 3 seconds with a altitude change of 53 meters. The velocity changes from 112.7 m/s to 7.1 m/s over that interval. Will a linear model fit this data? Why or why not? Find a second-order (quadratic polynomial) to fit this data.

Acknowledgements